Analytic Expressions for Neoclassical Transport Coefficients Including Finite Banana-Width Effect around the Magnetic Axis

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Abstract
A simple interpolation formula between the neoclassical transport coefficients at the magnetic axis and the conventional ones is presented for an axisymmetric toroidal plasma. Using this formula, the analytic expressions for the parallel current and the radial particle and heat fluxes are derived. In addition, the bootstrap current due to fusion-produced alpha particles is also obtained. Our results reduce to the conventional ones when the ratio $\delta_1/\sqrt{\varepsilon}$ is much less than unity, where $\varepsilon$ is the inverse aspect ratio, and $\delta_1 = 2q_\alpha \rho_a / k R_0$ with the major radius $R_0$, the elongation parameter $k$, the safety factor $q_\alpha$ at the magnetic axis, and the Larmor radius $\rho_a$ for species $a$. The finite banana-width effect is shown to modify the conventional neoclassical transport coefficients around the magnetic axis satisfying the condition $\sqrt{\varepsilon} < \delta_1^3$.

Keywords: neoclassical transport coefficient, finite banana-width, magnetic axis, alpha particle

1. Introduction
In the neighborhood of magnetic axis, a particle orbit \[1,2\] is quite different from the conventional banana orbit and a thin-banana approximation in the standard neoclassical transport theory \[3-5\] is broken down. The revision of the standard theory around the magnetic axis has been studied by many authors, and several expressions for the transport coefficients at the magnetic axis are presented \[6-14\]. Recently, Wang \[15\] proposed a simple theory for estimating the parallel current at the magnetic axis. In his theory, the parallel current is obtained from the conventional theory only by replacing the conventional trapping condition with the trapping condition at the magnetic axis. More recently, Helander \[16\] discussed the neoclassical transport at the magnetic axis, and claimed that this transport cannot be described independently of the sources of particles and heat in the low collisionality regime.

In order to use the theoretically obtained transport coefficients at the magnetic axis in a practical problem such as the calculation of bootstrap current, we need to connect these transport coefficients to the conventional ones in the region away from the axis. Making an attempt to derive this connection theoretically is a formidable problem, and thus now, simple phenomenological interpolation formulae are proposed \[6,7,12\]. The purpose of this paper is to present a more sophisticated interpolation formula between the neoclassical transport coefficients at the magnetic axis and the conventional ones. We take into account the finite banana-width effect at the magnetic axis by employing the Wang’s method \[15\]. However, in the Wang’s calculation, the collision operator is approximated by the pitch-angle scattering term although more

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general collision operator including the momentum conserving term is usually used in the conventional neoclassical transport theory. Therefore, including the momentum conserving collision term in the Wang's calculation, we derive the distribution function at the magnetic axis. The resulting distribution function takes quite similar form of the distribution function in the conventional neoclassical transport theory. This similarity leads to a natural interpolation formula between these two distribution functions.

This paper is organized as follows. In Sec.2, a simple trapping condition in velocity space is proposed. Using this condition, we obtain an interpolated distribution function in the low collisionality regime in Sec.3. In Sec.4 and 5, we derive the analytic expressions for the parallel current and the radial fluxes. In Sec.6, applying our method to the alpha-induced bootstrap current, we show that the conventional alpha-induced bootstrap current is significantly modified around the magnetic axis by the finite banana-width effect. The concluding remarks are shown in Sec.7.

2. Trapping Boundary

Let us consider a tokamak configuration

$$B = I (\psi) \nabla \phi + \nabla \phi \times \nabla \psi,$$

where $$\psi$$ is the poloidal flux and $$\phi$$ is the toroidal angle around the magnetic axis. The trajectory of a charged particle in this configuration is determined by using the constants of energy $$E = m_v v^2/2 + e_v \phi(\psi),$$ magnetic moment $$\mu = v_\perp^2/(2B),$$ and canonical angular momentum $$P = \psi - I v_\parallel \Omega_a,$$ where $$v_\parallel = v \cdot b$$ with $$b = B/B, v/v_\parallel = \sqrt{\mu^2 - v_\perp^2}, \phi$$ is the equilibrium electrostatic potential, and $$m_v, e_v$$ and $$\Omega_a$$ are the mass, charge and Larmor frequency for species a. The boundary in velocity space for a trapped particle that reverses the sign of $$v_\parallel$$ was discussed in detail by Chu [2], and a little bit complicated trapping boundary was obtained. However, we here use an approximate trapping boundary instead of the exact one in order to go further analytically. It should be noted here that our definition of trapped particles is different from that of potato particles [8].

We approximate the poloidal flux as $$\psi = k B_0 \eta^2/(2q_0)$$ around the magnetic axis, where $$k$$ is the elongation parameter, and $$q_0$$ and $$B_0$$ are the safety factor and the magnitude of magnetic field at the magnetic axis. Then, by using three constants of motion, a particle with the parallel velocity $$v_\parallel$$ at a radial position $$r$$ and a poloidal angle $$\theta = \pi/2$$ is shown to reverse the sign of parallel velocity when the conditions

$$\left| \frac{v_\parallel}{v} \right| < \xi_+ \quad \text{for } e_v v_\parallel > 0,$$

$$\left| \frac{v_\parallel}{v} \right| < \xi_- \quad \text{for } e_v v_\parallel < 0$$

are satisfied (Appendix A). The parameters $$\xi_+ (> 0)$$ and $$\xi_- (< 0)$$ are two real roots of the quartic equation

$$\xi^4 + 6\xi^2 - 8\xi^2 = 0,$$

where the inverse aspect ratio $$\epsilon = r/R_0$$ with the major radius $$R_0,$$ and $$\delta_a = 2q_0 \rho_s/kR_0$$ with $$\rho_s = \rho/[\Omega_a].$$ The explicit forms of these $$\xi$$ are written as [17]

$$\xi_{\pm} = \frac{1}{2} \left( -\sqrt{u \pm \sqrt{u^2 + 4u^2 + 4\epsilon^2}} \right),$$

where $$u = s_+ + s_-$$ and

$$s_{\pm} = \frac{1}{2} \left( \delta^2 \pm \sqrt{\left( \frac{4}{3} \epsilon^2 \right)^2 + \frac{1}{4} \delta^4} \right)^{1/3}.$$

The trapping boundary (1) in velocity space for a positively charged particle is shown for $$\delta_a = 0.03$$ and $$\epsilon = 0.1$$ in Fig. 1. The dotted lines in this figure represent the conventional trapping boundary. We rewrite the trapping condition in terms of a variable $$\lambda = (1 - v_\parallel^2/v^2)/B$$ and propose the following approximate trapping condition

$$\frac{1}{B_{\min}} > \lambda > \frac{1 - \xi_+^2}{B_0} \quad \text{for } e_v v_\parallel > 0,$$

$$\frac{1}{B_{\min}} > \lambda > \frac{1 - \xi_-^2}{B_0} \quad \text{for } e_v v_\parallel < 0.$$
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The limiting forms for \( \xi \) are given as follows: \( \xi \rightarrow \pm \sqrt{\varepsilon} \) when \( \sqrt{\varepsilon} \gg \delta \), and \( \xi \rightarrow 0 \) and \( \xi \rightarrow -\delta \) when \( \sqrt{\varepsilon} \ll \delta \). Therefore, the trapping condition (5) reduces to the conventional condition \( 1/B_{\text{tor}} \geq \lambda \geq 1/B_{\text{max}} \) in the region where \( \sqrt{\varepsilon} \gg \delta \); it reduces to the trapping condition at the magnetic axis [15]

\[
\left| \frac{v_i}{\mu} \right| < \delta, \quad \text{and} \quad e_i v_i < 0.
\]

3. Distribution Function

First, we summarize the Wang’s method of calculating the distribution function at the magnetic axis. Let us start with a drift kinetic equation

\[
\partial_t f + \nu \frac{\partial f}{\partial \nu} = C(f),
\]

where \( C \) is the collision operator and the partial derivative is taken at fixed \((E, \mu, \sigma)\). We write the distribution function in the form: \( f = f_0 + f_1 \), with \( f_1 = -i \nu f_0 \), \( f_0 = g(P, E, \mu, \sigma) \), where \( \sigma = v_i \nu v_i f_0 = \partial f_0 / \partial \nu \), and \( f_0 \) is the local Maxwellian distribution function. Then the perturbed function \( g \) at the low collisionality regime is determined from the solubility condition

\[
\left\langle C(g) \right\rangle = \left\langle C \left( \frac{v_i}{\Omega_a f_0} \right) \right\rangle = 0,
\]

where we have used the relation \( \left( \partial_t \partial \theta + \nu \partial \nu \partial \nu \right) f = \partial \nu \partial \nu \partial \nu \), and define the time average operator \( \left\langle A \right\rangle = \frac{1}{\varepsilon} \int \frac{\partial \theta \partial \theta}{\partial \theta} A \)

with the integral taking along the guiding center orbit and the bounce period \( \tau = \varepsilon / \partial \theta \).

When we approximate the collision operator by the pitch-angle scattering form, the solubility condition becomes

\[
\nu \frac{v_i}{\partial \mu} \frac{\partial g}{\partial \mu} = -i \nu \Omega_a f_0 \left( \frac{v_i}{\Omega_a} \right),
\]

where \( v \) is the collision frequency. Wang proposed the approximation \( \left( \frac{v_i}{\Omega_a} \right) \rightarrow 0 \) for the trapped particles that reverse the sign of their \( v_i \) along the drift orbit. This approximation leads to

\[
\frac{\partial g}{\partial \mu} = 0.
\]

For passing particles, neglecting the toroidal effect, he proposed the following solution

\[
\frac{\partial g}{\partial \mu} = -B I \frac{1}{\Omega_a} v_i f_0.
\]

Therefore, noticing the trapping condition (6) at the magnetic axis, we obtain the perturbed function \( g \) at the axis from (11) and (12)

\[
g_a = \frac{\sigma}{2} H(\lambda_{\text{axis}} - \lambda) \int_0^{\lambda_{\text{axis}}} \frac{\partial \lambda \nu B}{1 - \lambda B} K_a(v_i),
\]

where \( \lambda_{\text{axis}} = (1 - \delta^2) B_0 / e, \nu v_i < 0 \) and \( \lambda_{\text{axis}} = 1 / B_0 \) for \( e_i v_i > 0 \) and \( H(x) \) is the Heaviside step function.

We here introduce an approximate form for the linearized collision operator \( C_{ab}(f_a, f_b) \), where \( C_{ab}(f_a, f_b) \) and \( C_{ab}(f_a, f_b) \) represent the Landau collision operators for the perturbed test and field particles. This approximate form is written as [18]

\[
C_{ab}(f_a, f_b) = v_i \left( \frac{1}{B} \frac{1}{\Omega_a} v_i \frac{\partial f_0}{\partial \mu} \frac{\partial f_0}{\partial \mu} \right),
\]

where \( \xi = \nu v_i / v_i \). The fraction of trapped particles, \( f_{\text{trapped}}(v_i) \), is roughly equal to that of potato particles [8].

In the standard neoclassical transport theory, a
perturbed distribution function $g_\Omega$ can be written in the banana regime as \[4,18\]
\[g_\Omega = \frac{\sigma}{2f_\Omega} H(\lambda - \lambda) \int_0^{\lambda_c} \frac{d\lambda}{\sqrt{1 - \lambda B}} K_\Omega(v), \] (18)
where $\lambda_c = 1/B_{\text{max}}$, $\langle \cdot \rangle$ denotes a flux surface average, and
\[f_\Omega = \frac{3}{4} \langle B^2 \rangle \int_0^{\lambda_c} \frac{\lambda d\lambda}{\sqrt{1 - \lambda B}}. \] (19)
The function $K_\Omega$ in (18) is determined from the equation
\[\frac{f_\Omega}{f_\Omega} v^2_\Omega K_\Omega (\lambda, v) - \frac{\varepsilon}{f_\Omega} K_\Omega (\lambda, v), \] (20)
where $f_\Omega = 1 - f_c$ is the fraction of trapped particles.
Comparing (15) and (17) with (18) and (20), we propose the following interpolated perturbed distribution function
\[g_\Omega = \frac{\sigma}{2f_\Omega} H(\lambda' - \lambda') \int_0^{\lambda_c} \frac{d\lambda}{\sqrt{1 - \lambda B}} K_\Omega(v), \] (21)
where $\lambda' = \lambda^*_1$ for $e_\Omega v_\Omega > 0$ and $\lambda' = \lambda^*_2$ for $e_\Omega v_\Omega < 0$. The fraction of passing particles is defined by
\[f_c^* = \frac{3}{8} B^2 \left( \int_0^{\lambda_c} \frac{\lambda d\lambda}{\sqrt{1 - \lambda B}} + \int_0^{\lambda_c} \frac{\lambda d\lambda}{\sqrt{1 - \lambda B}} \right), \] (22)
where we use the approximation $\sqrt{1 - \lambda B} \sim \sqrt{1 - \lambda B_0}$ for $\lambda > \lambda_c$. The function $K_\Omega$ in (21) is determined by the equation
\[\frac{f_\Omega}{f_\Omega} v^2_\Omega K_\Omega (v) = \int_0^{\lambda_c} \frac{\lambda d\lambda}{\sqrt{1 - \lambda B}} K_\Omega (v), \] (23)
with $f_c^* = 1 - f_c^*$. The distribution function (21) reduces to the expression (15) at the magnetic axis, and it also reduces to the expression (18) in the region $\sqrt{\varepsilon} \gg \delta_0^3$.

The equation (23) for $K_\Omega$ can be solved by the moment approach. In this approach, we expand the function $K_\Omega$ in the form
\[K_\Omega = \frac{m_\Omega}{T_\Omega} \left( B^2 \right) v \left( u_{\Omega \theta} - \frac{2}{5} \frac{\varepsilon^2}{v_\Omega^2} \right) P_\Omega f_\Omega. \] (24)
The poloidal flows $u_{\Omega \theta}$ and $q_{\Omega \theta}$ in (24) are determined from a coupled algebraic equations that are derived by inserting (24) into (23) and taking moments with respect to $v^2$ and $v^2 (v^2/v_\Omega^2 - 5/2)$.

4. Bootstrap and Ohmic Currents

Using the distribution function (21), we can express the bootstrap current $J_B$ and Ohmic current $J_E$ in the form:
\[J_B = - \frac{L_{\text{Bo}}}{B} \int_{T} \left( \frac{P_\Omega}{P_\theta} + \frac{1}{Z} \frac{T_\Omega}{T_\theta} \right) \left( L^{(B)}_1 + L^{(B)}_2 + L^{(B)}_3 \right) T^2_\Omega T^4_\Omega, \] (25)
\[J_E = \varepsilon^2 n_\Omega \mu_{\Omega \theta} L^{(B)} \left( \frac{B E_{\Omega \theta}}{B} \right) \] (26)
where $\tau_{\Omega B} = 3(1/4(x_{\Omega B})).$ and $Z$ is the ion charge number. The transport coefficients $L^{(B)}_1$ and $L^{(B)}_2$ are given by
\[L^{(B)}_1 = \frac{1}{D_\Omega} \left( \mu_\Omega + l_3 \right) \frac{\mu_{\Omega \theta} + \mu_{\Omega \Omega} + l_3 \mu_{\Omega \theta}}{l_3 \mu_{\Omega \theta} + l_3 \mu_{\Omega \Omega} + l_3 \mu_{\Omega \Omega} + l_3 \mu_{\Omega \Omega}} \] (27)
and $L^{(B)}_2 = \frac{1}{D_\Omega} \left( \mu_{\Omega \theta} + l_3 \right) \frac{\mu_{\Omega \theta} + l_3 \mu_{\Omega \theta}}{l_3 \mu_{\Omega \theta} + l_3 \mu_{\Omega \theta} + l_3 \mu_{\Omega \theta} + l_3 \mu_{\Omega \Omega}}.$ The viscosity coefficients $\mu_{\Omega \Omega}$ are defined by
\[\mu_{\Omega \Omega} = \frac{\tau_{\Omega B} 8\pi n_\Omega}{3}, \] (28)
with $\tau_{\Omega B} = 3\sqrt{\nu_\Omega^3/(4\nu_\Omega)}.$ These viscosity coefficients $\mu_{\Omega \Omega}$ reduce to the conventional ones $\mu_{\Omega \Omega}$ for $\sqrt{\varepsilon} \gg \delta_0^3$, and these coefficients on the magnetic axis become
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\[
\begin{bmatrix}
\mu_{e*} \\
\mu_{i*} \\
\mu_{n*}
\end{bmatrix} = \varepsilon^{1/3} \frac{2\pi \tau_{ee}}{n_s} \left( \begin{array}{c}
1 \\
\frac{v_p^2 - 5/2}{(v_p^2 - 5/2)^{3/2}}
\end{array} \right)
\times \int_0^\infty dv \left( \frac{v}{v_p} \right)^{\alpha} \frac{\nu^4}{v_0^3} \nu^3(v) f_{s0} \left( \frac{v^2}{v_p^2} - 5/2 \right)^{3/2} .
\] 

(29)

for \( \varepsilon^{1/3} \ll 1 \), where we have redefined the parameter \( \delta \) by using the normalized Larmor radius \( \rho_s = v_p/\Omega_s \) instead of \( v_p/\Omega_s \), and we will use this redefined \( \delta \) from now. We note here that the effect of finite banana width on the transport coefficients for the bootstrap current and the Ohmic current is taken into account through the viscosity coefficients \( \mu_{*} \).

In Figs. 2 and 3, the electron viscosity coefficients \( \mu_{e*} \) for \( \delta_e = 0.001 \) and \( Z = 1 \), and the ion viscosity coefficients \( \mu_{i*} \) for \( \delta_i = 0.05 \) in a tokamak with circular cross section are depicted as a function of the inverse aspect ratio \( \varepsilon \). These figures show that the viscosity coefficients deviate from the conventional ones shown in dotted curves within \( \varepsilon < \delta_e^{1/3} \). In Fig. 4, the transport coefficient \( L_{L1}^{(B)} \) for the bootstrap current is plotted as a function of \( \varepsilon \) for \( \delta_e = 0.001 \). Comparing the conventional transport coefficient plotted in dotted curve, we find that the bootstrap current is modified very near the magnetic axis and it remains finite on the axis when \( \rho_i/B_0 \) (or \( T_i/B_0 \)) with the poloidal magnetic field \( B_0 \) is finite on it.

5. Radial Fluxes

When we take into account the finite banana-width effect, the radial components of particle and heat fluxes nonlocally depend on the radial gradients of the equilibrium pressure, temperature and electrostatic potential. Therefore, according to Shaing et al. [8], we here define the radial particle and heat fluxes as

\[
\begin{bmatrix}
\rho_{\psi*} \\
q_{\psi*}/T_s
\end{bmatrix} = \frac{1}{\Delta \psi} \int d\psi \left( \begin{array}{c}
\frac{1}{\Delta \psi} \int d\psi \nabla \psi \left( \frac{1}{v^2/v_p^2 - 5/2} \right) f_{s0} \left( \frac{v}{v_p} \right)^{\alpha} \nu^3(v) f_{s0} \left( \frac{v^2}{v_p^2} - 5/2 \right)^{3/2} \\
\frac{1}{\Delta \psi} \int d\psi \nabla \psi \left( \frac{1}{v^2/v_p^2 - 5/2} \right) f_{s0} \left( \frac{v}{v_p} \right)^{\alpha} \nu^3(v) f_{s0} \left( \frac{v^2}{v_p^2} - 5/2 \right)^{3/2}
\end{array} \right) .
\]

(30)

where \( \nabla \psi \), \( \nu \) is the radial component of drift velocity and \( \Delta \psi \) is the typical orbit width. These radial fluxes can be expressed in the form [8]

\[
\begin{bmatrix}
\rho_{\psi*} \\
q_{\psi*}/T_s
\end{bmatrix} = \frac{1}{\Delta \psi} \int d\psi \left( \begin{array}{c}
\int dv \nabla \psi \left( \frac{1}{v^2/v_p^2 - 5/2} \right) f_{s0} \left( \frac{v}{v_p} \right)^{\alpha} \nu^3(v) f_{s0} \left( \frac{v^2}{v_p^2} - 5/2 \right)^{3/2} \\
\int dv \nabla \psi \left( \frac{1}{v^2/v_p^2 - 5/2} \right) f_{s0} \left( \frac{v}{v_p} \right)^{\alpha} \nu^3(v) f_{s0} \left( \frac{v^2}{v_p^2} - 5/2 \right)^{3/2}
\end{array} \right) .
\]

(31)

The radial fluxes (31) reduce to the conventional flux-
The transport coefficient $L_i^{(0)}$ for the bootstrap current versus the inverse aspect ratio $\epsilon$ for $\delta_i = 0.001$ and $Z = 1$. The dotted curve represents the conventional transport coefficient.

Fig. 4

Surface averaged radial fluxes in the region far from the magnetic axis.

Similar to the parallel current, we estimate the effect of finite banana-width near the magnetic axis by replacing the conventional viscosity coefficients $\mu_{ak}$ with newly defined ones $\mu_{ak}^*$. Then the radial fluxes are decomposed into the neoclassical fluxes and the Pfirsch-Schlüter fluxes that agree with the conventional ones. The neoclassical radial fluxes $F_{e}^{(N)}$ and $q_{e}^{(N)}$ are expressed in the form

$$F_{e}^{(N)} = Z \frac{\Gamma_{e}^{(N)}}{B} \int v^3 v_{\perp} K_{e} \frac{f_{e}}{f_{e}^*} \, dv$$

$$q_{e}^{(N)} = \frac{IB}{\Omega_{e}} \frac{1}{B^2} \frac{4\pi}{3} \int v^3 v_{\perp} K_{e} \frac{f_{e}}{f_{e}^*} \left( \frac{v^2}{v_{\perp}^2} - \frac{5}{2} \right) \, dv$$

Solving (32) with (33), we can obtain the poloidal flows $u_{e\theta}$ and $q_{e\theta}$.

$$u_{e\theta} = \frac{1}{B} \frac{I_{c} T_{e}}{e D_{e}}$$

We show the explicitly calculated transport coefficient $L_i = \sqrt{2}(\mu_{e}^{(1)} - \mu_{\Omega}^{(1)} \mu_{e}^{(1)})/D_i$ for the radial heat flux $q_{e}$ in Fig. 5. This figure shows the modification of this transport coefficient near the magnetic axis for $\delta_i = 0.05$.

The transport coefficient $L_i$ for the radial ion heat flux can be written as $L_i = -0.55 \delta_i^{1/3}$ at the magnetic axis when $\delta_i^{1/3} \ll 1$. Several expressions for this transport coefficient are obtained [6-8, 14], but these expressions differ by a numerical factor of order unity. For instance, Shaing et al. [8] derived the expression as $L_i = -1.7611^{1/3}$. The precise numerical factor will be determined by future numerical studies.

6. Bootstrap Current Due to Alpha Particles

Goloborod’ko et al. [19] investigated the finite banana-width effect on the bootstrap current due to the fusion-produced alpha particles, and showed that these alpha particles can produce a current at the magnetic axis. In their calculation, they took into account only the slowing-down of the alpha particles by electrons in Coulomb collisions. This approximation for the collision term is valid in the regime $v_{\perp}/v_{\|} < 1$, where $v_{\perp}$ is the birth velocity of alpha particle and $v_{\|}$ is the critical velocity defined as $v_{\|} = (3\sqrt{\pi}/4) \Sigma (m_{e} n_{e} e^2/m_{e} n_{e} e^2) v_{e}$.
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Fig. 5 The transport coefficient $L_i$ for the radial ion heat flux versus the inverse aspect ratio $\varepsilon$ for $\delta_i = 0.05$. The dotted curve represents the conventional transport coefficient.

with $\sum$ meaning a summation over only ion species. The ratio of the critical velocity to the birth velocity is estimated as $v_c/v_a \sim 0.1T_e$ [keV] $^{1/2}$, so that this ratio is not much less than unity for fusion plasma. Therefore, the importance of retaining not only the slowing-down drag but also the pitch-angle scattering in the collision term has been pointed out in the recent studies on the alpha-induced bootstrap current [20, 21]. Recently, Taguchi and Shioya [22] derived an analytic expression for the bootstrap current due to the alpha particles, which can be used for the regime $v\varepsilon/v_a > 0.2 - 0.3$. Explicit analytic expressions for these parameters are given in Appendix B. We here only note that the geometric parameters $A_i$ and $\kappa_i$ that are determined from the geometric structure of the equilibrium magnetic field are expressed in terms of coefficients $a_n(n = 1 - 3)$ and $b_n(n = 1, 2)$, defined by

$$a_n = \left( \frac{B}{\lambda} \right)^{n+1} \int_0^{\lambda_c} \frac{\lambda' d\lambda'}{\sqrt{\lambda - \lambda B}}; \tag{39}$$

$$b_n = \left( \frac{B}{\lambda} \right)^{n+1} \int_0^{\lambda_c} \lambda' \left( \frac{1}{\sqrt{1 - \lambda B}} \right) \times \left( \int_0^{\lambda_c} \frac{d\lambda'}{\sqrt{1 - \lambda B}} \right)^2 d\lambda'. \tag{40}$$

Replacing $\lambda_c$ by $\lambda_c^f$ in $a_n$ and $b_n$ and considering the expression (36) as

$$J_{ab} = \left( \frac{1}{2} \right) \left[ J_{ab}(\lambda_c \rightarrow \lambda_c^f) + J_{ab}(\lambda_c^f \rightarrow \lambda_c) \right]; \tag{41}$$

we can estimate the bootstrap current including finite trapping-ratio effect near the magnetic axis. In Fig. 6, the transport coefficient $L_{2a}$ for the alpha-induced bootstrap current is plotted as a function of $\varepsilon$ for $Z = 0.6$, $v\varepsilon/v_a = 0.5$, and $\delta_a = 0.03$ and 0.1, where $Z = \Sigma n_i e_i^2 \log A_i / (\Sigma n_i e_i^2 m_i \log A_i / m_i)$. Comparing with the dotted curve obtained by the conventional theory, we find the finite banana-width effect strongly modifies the standard bootstrap current near the magnetic axis.

7. Conclusions

We have presented the simple interpolation formula between the neoclassical transport coefficients at the magnetic axis and the conventional ones. According to this formula, the analytic expressions for the parallel current and the radial particle and heat fluxes are

$$L_{2a}^{(B)} = \int f_i \delta_i \sum A_i \left[ 1 - \frac{1}{\kappa_i} \right] p_i' \tag{38},$$

where $n_a$ is the birth rate of alpha particles and $\tau_i = 3m_i m_a e^2 / (16\pi e^2 e_a^2 n_i \ln A)$ is the slowing-down time. The infinite series in the bootstrap current converges rapidly except for the small $v\varepsilon/v_a$ limit, and thus the first few terms in this series suffice to provide the accurate result for the bootstrap current in our practically interesting parameters. Taguchi and Shioya [22] have approximated the infinite series by the first two terms, and derived simple expressions for the parameters $s_1, s_2, A_i, \kappa_i, p_i$ and $\delta_i (i = 1, 2)$ by using Padé approximation and a variational method. The bootstrap current obtained by using these simple expressions agrees quite well with the numerically calculated exact one for $v\varepsilon/v_a > 0.2 - 0.3$. Explicit analytic expressions for these parameters are given in Appendix B. We here only note that the geometric parameters $A_i$ and $\kappa_i$ that are determined from the geometric structure of the equilibrium magnetic field are expressed in terms of coefficients $a_n(n = 1 - 3)$ and $b_n(n = 1, 2)$, defined by

$$a_n = \left( \frac{B}{\lambda} \right)^{n+1} \int_0^{\lambda_c} \frac{\lambda' d\lambda'}{\sqrt{1 - \lambda B}}; \tag{39}$$

$$b_n = \left( \frac{B}{\lambda} \right)^{n+1} \int_0^{\lambda_c} \lambda' \left( \sqrt{1 - \lambda B} \right) \times \left( \int_0^{\lambda_c} \frac{d\lambda'}{\sqrt{1 - \lambda B}} \right)^2 d\lambda'. \tag{40}$$

Replacing $\lambda_c$ by $\lambda_c^f$ in $a_n$ and $b_n$ and considering the expression (36) as

$$J_{ab} = \left( \frac{1}{2} \right) \left[ J_{ab}(\lambda_c \rightarrow \lambda_c^f) + J_{ab}(\lambda_c^f \rightarrow \lambda_c) \right]; \tag{41}$$

we can estimate the bootstrap current including finite trapping-ratio effect near the magnetic axis. In Fig. 6, the transport coefficient $L_{2a}$ for the alpha-induced bootstrap current is plotted as a function of $\varepsilon$ for $Z = 0.6$, $v\varepsilon/v_a = 0.5$, and $\delta_a = 0.03$ and 0.1, where $Z = \Sigma n_i e_i^2 \log A_i / (\Sigma n_i e_i^2 m_i \log A_i / m_i)$. Comparing with the dotted curve obtained by the conventional theory, we find the finite banana-width effect strongly modifies the standard bootstrap current near the magnetic axis.
obtained for tokamak plasmas. Our analytic expressions reduce to the conventional ones when the ratio $\delta B/\delta e$ is much less than unity. The finite banana-width effect modifies the conventional neoclassical transport coefficients around the magnetic axis satisfying the condition $\sqrt{e} < \delta e$. The break-down of the standard neoclassical transport theory is significant for high energy particles. The alpha-induced bootstrap current obtained by the conventional theory is shown to be strongly modified around the magnetic axis.

Various approximations are needed in order to analytically calculate the finite banana-width effect on the neoclassical transport coefficients. Although the several expressions for the transport coefficients at the magnetic axis are obtained, the numerical factors differ in these expressions. The determination of precise numerical factors seems to be beyond the scope of analytic treatment. Its determination is left for future numerical studies.

The time-derivative $\partial f_p/\partial t$ is usually neglected in the drift kinetic equation since the time scale of the cross-field transport is much larger than that of establishing a local equilibrium. Recently, Helander [16] pointed out that a separation of these two time scales is no longer satisfied near the magnetic axis and the time-derivative can not be neglected in the drift kinetic equation. Therefore, in order to consider the steady-state problem, we must add the source term such as particle and heat sources to the drift kinetic equation near the magnetic axis. The particle and heat sources, which are assumed to be comparable to the collision term in magnitude, do not change the transport coefficients for the radial fluxes and the parallel current within the framework of the conventional theory for the banana regime. However, in taking the finite banana-width effect into consideration, an additional perturbed distribution function driven by the sources may induce the radial fluxes and the parallel current. Investigation of this source-induced transport, which is a formidable problem as is pointed out by Helander, is also left for future study.

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Appendix A
Let us consider a particle with a pitch-angle parameter $\xi = \theta/\theta_0$ at a radial position $r$ and a poloidal angle $\theta = \theta_0$ on the trajectory. Setting $\theta_0 = \pi$ and $\xi_0 = 0$ in (42) and (43), we can obtain the trapping condition (1) in velocity space for a trapped particle that has toroidal turning point in its trajectory. This simply estimated trapping condition is not exact for a particle with $e_v < 0$ since the pitch-angle parameter $\xi_0$ does not become zero when $\theta_0 = \pi$. The exact trapping condition for $e_v < 0$ is derived from the equations (42) and (43) with $\theta_0 = \pi$, and the condition $d\xi/dr = 0$ instead of $\xi_0 = 0$ [2]. However, the exact boundary parameter $\xi_0$ for $\xi_0 < 0$ is well approximated by $\xi_0$ derived by setting $\xi_0 = 0$. This is because the pitch-angle parameter $\xi_0$ at $\theta = \pi$ nearly equals to zero although it does not become exactly zero. The numerically calculated boundary parameter $\xi_0$ is depicted by dashed curve in Fig. 1.

Appendix B
The geometric parameters $\kappa_1$ and $\Lambda_1$ are expressed in terms of the coefficients $a_n(n = 1 - 3)$ and $b_n(n = 1, 2)$ in the form:
Analytic Expressions for Neoclassical Transport Coefficients Including Finite Banana-Width Effect

\[ \kappa_1 = \frac{\beta - \sqrt{\beta^2 - \alpha \gamma}}{\alpha} , \quad \kappa_2 = \frac{\beta + \sqrt{\beta^2 - \alpha \gamma}}{\alpha} , \quad (44) \]

\[ A_i = \frac{1}{2} \left( \frac{B_i^2}{B^2} \right) \times \frac{(a_1 + \frac{1}{2} \delta_i a_2)^2}{2(a_1^2 - 2a_1) + 4(-a_2 + b_0) \delta_i + 4(-\frac{a_3}{3} + b_1) \delta_i} , \quad (45) \]

where

\[ \alpha = 4 \left[ (a_0^2 - 2a_1) \left( 2b_1 - \frac{2}{3} a_3 \right) - (b_0 - a_2)^2 \right] , \quad (46) \]

\[ \beta = 4a_1 \left( b_1 - \frac{a_3}{3} \right) + 2 \left( b_1 + \frac{a_3}{3} \right) (a_0^2 - 2a_1) - 2a_2(b_0 - a_2) , \quad (47) \]

\[ \gamma = 4a_1 \left( b_1 + \frac{a_3}{3} \right) - a_2^2 , \quad (48) \]

\[ \delta_i = \frac{2}{2 \kappa_i} \left[ a_1 - \kappa_i (a_0^2 - 2a_1) \right] , \quad (49) \]

The parameters \( s_1, s_2, p_1 \) and \( p_2 \) in the transport coefficients (37) and (38) are given by

\[ \begin{bmatrix} s_1 \\ s_2 \end{bmatrix} = \begin{bmatrix} 1/6 - J(y_e) / 3 \\ -y_e^3 / [9(1+y_e^3)] + 2J(y_e) / 9 \end{bmatrix} , \quad (50) \]

\[ p_1 = \frac{1}{2} - J(y_e) - \frac{12}{(\tilde{Z} \kappa_i + 4) \left( \tilde{Z} \kappa_i + 3 \right)} [J(y_e) - J(y_e^c)] , \quad (51) \]

\[ p_2 = -\frac{4}{\tilde{Z} \kappa_i + 3} [J(y_e^c) - J(y_e)] + \frac{8}{(\tilde{Z} \kappa_i + 3) \left( \tilde{Z} \kappa_i + 4 \right)} \left[ \tilde{Z} \kappa_i + 9 \right] J(y_e) - 6J(y_e^c) - \frac{\tilde{Z} \kappa_i + 3}{2} \frac{y_{e}^3}{1 + y_{e}^3} \right] , \quad (52) \]

where

\[ y_e = v_e / v_0 , \quad y_e^c = y_e \left( (\tilde{Z} \kappa_i + 7) / 4 \right)^{1/2} , \quad (53) \]

\[ J(y_e) = y_e^3 \left[ 1 \log \left( \frac{1 + y_e^c}{(1 + y_e^c)^3} \right) + \frac{1}{\sqrt{3}} \tan^{-1} \left( \frac{2 - y_e}{\sqrt{3} y_e} \right) + \frac{\pi}{6 \sqrt{3}} \right] . \quad (54) \]

References