Abstract

Review is devoted to MHD theory of equilibrium and stability of a plasma in "conventional" stellarators, which are the systems with a planar circular axis and with helical fields. The possibility to describe a plasma in these systems by the two-dimensional reduced equations is discussed and solutions of typical equilibrium and stability problems found on the basis of these equations are analyzed. The next problems are considered here: fixed- and free-boundary plasma equilibrium, effect of the vertical and quadrupole fields on the stellarator configuration, the problem of limiting plasma pressure in stellarators, problems of magnetic diagnostics, plasma stability with respect to Mercier, ballooning and large-scale modes. Theory of plasma equilibrium is presented with illustrative analytical calculations in traditions of well established "classical" tokamak theory. General features of finite-pressure plasma behavior in tokamaks and stellarators are compared and principal differences are pointed out. In the section devoted to the plasma stability the main equations are presented and numerical results are discussed.

Keywords:
MHD, stellarator, plasma equilibrium, plasma stability, finite-β effects, magnetic diagnostics,
11.1. Determination of plasma shift from magnetic measurements
11.2. Theory of diamagnetic measurements in stellarators
12. MHD plasma stability in stellarators
12.1. Sufficient stability criterion
12.2. Mercier criterion for stellarators
12.3. Ballooning mode stability
12.4. Stability of nonlocal (low-n) modes
13. Conclusion

1. Introduction

For stellarator studies which endured upsurges and recessions in their evolution the last decade was a period of fast and significant progress. Suffice it to mention among the principal events the design studies, construction and starting of operation of the "new generation" helical devices ATF [11] and W7-AS [2]. These machines together with Japanese installation Heliotron E [3] are now the largest and most equipped stellarators. Each of them is unique in its own way which is often emphasized in the very definition of the device type: torsatron, modular stellarator, heliotron. The word "stellarator" will be used for them here as having the generally accepted wide meaning: a helical system. Stellarators (in the narrow sense of this word), torsatrons and heliotrons are unified by this word, which symbolizes the affiliation of the systems to a certain class, when we speak about physics but not about construction of the device.

The history of stellarators, their advances and failures are described in details in a series of excellent reviews [4-21]. Without reiteration of what was written there, let us note only one (more or less reflected in [1-21]) essential circumstance which determined the subject of the present article: from the very beginning of stellarator studies the stellarator concept, concrete physical programs developed and evolved under the strong (in the last decade, maybe, decisive) influence of MHD theory predictions.

Stellarators are the toroidal systems in which hot plasma is contained by a magnetic field created by external currents. The problem how to make a "good" stellarator is reduced to the problem of the choice of vacuum magnetic configuration. For the complete solution of these problems it is necessary to analyze the transport and MHD equilibrium and stability of a plasma. Some idea about the influence of the both research areas on the conceptual ideology of stellarators can be illustrated by the next facts from their history.

Stellarators have been contrived by L. Spitzer, Jr., in 1951 as closed systems with twisted spatial axis looking like figure eight [22,4]. Magnetic field lines in such solenoid are twisted around its axis, and that's enough, in principle, for good confinement of charged particles. Simplicity and elegance of this idea have been quickly appraised by U.S. physicists: the first "Spitzer's figure-8" with magnetic field of up to 1 kG, opening the list of stellarators, appeared already in 1952 [4]. Though particle diffusion was higher than it was expected, the whole experimental results were encouraging. But soon after this an outlined extensive research program based on optimistic expectations had to be reconsidered. The main cause was the plasma instability. Started in 1954 theoretical studies of MHD plasma stability in stellarators not only explained this phenomena, but also prompted the means of its suppression: the shear of magnetic field lines. To create configurations with a shear it was proposed in 1956 to use helical current-carrying conductors [22]. This second concept proposal of L. Spitzer allowed at the same time to simplify stellarator configuration: with helical windings it could have a planar circular axis. Thus the results of MHD analysis led to the change of the stellarator appearance. Stellarators with planar circular axis and helical fields, which became finally the main (but not the only) object of stellarator studies, are called usually "conventional".

The necessity of attainment of high \( \beta \) (ratio of plasma and magnetic pressures) in fusion systems was recognized from the very beginning of fusion studies. But just the first experiments and after that theoretical estimates have clearly shown that in some cases even at low \( \beta \) plasma can be unstable. From that times one of the main problems of MHD theory, which is actual up to now, was a search for opportunities to increase \( \beta \). Soon after discovering the stabilizing role of shear, another means of instability suppression was found: magnetic well [23-28]. It is a more subtle characteristic of a
magnetic field than shear, and certain skill is needed for its creation. As it have been later shown by the theory, the systems without magnetic well have no prospects. Practically it led to more strict criteria of selection of stellarator configurations.

Interaction of a plasma with a confining field occurs because of the so-called equilibrium currents flowing through the plasma. The larger $\beta$, the stronger these currents. Their magnetic field distorts the magnetic configuration. Thus, $\beta$-rise is permissible only up to some critical value. Aspiration to increase this limit has led to the development of two very unlike concepts of advanced stellarators, which are represented by the ATF torsatron [1] and W VII-AS modular stellarator [2], and to the revival of interest to the spatial-axis systems (H-I in Australia [29-31] and TJ-II in Spain [31,32]). ATF is the conventional stellarator (torsatron) with shear and magnetic well and with a potential for shaping plasma column (including that during the discharge). This additional degree of freedom not only extends the operational range of the device, but also, theoretically, allows to improve significantly many plasma characteristics. Stellarators with changeable geometry of magnetic configuration, it is a new and, probably, necessary step to high $\beta$'s. The alternative is the last stellarator of Wendelstein series, W VII-AS [2,10], and W VII-X [33,34] which project is under study. They are almost shearless systems with complicated non-planar magnetic coils. They are interesting not only by their design. It seems that in W VII-AS, and all the more in W VII-X project, the requirements for $\beta$-increase and transport improvement are simultaneously satisfied better than in other systems. Let us emphasize that successful operation of modern stellarators Heliotron E, ATF, W VII-AS and CHS is the result of the profound elaboration of their projects which the main part is MHD calculations.

For more than 40 years of fusion researches MHD theory turned into a large science, thus to make its complete review is an impossible task. In the present review we pursue a more pragmatic objective: to give a brief and as simple as possible account of the main ideas and results of MHD theory of plasma equilibrium and stability in conventional stellarators. For a long time, while stellarators have been operated with low-$\beta$ plasma, there were no serious difficulties with ensuring the plasma equilibrium. In 1980s, when the attainment of high $\beta$'s became a real goal, the need appeared in developing adequate methods for the finite-$\beta$ plasma description in stellarators and reexamination of some obsolete ideas. It was necessary for the design studies of new installations and for planning experiments where finite-$\beta$ effects on plasma confinement could be essential. The complexity of the problems which became actual was that plasma self-fields could not be considered then as small against the background of external magnetic field, but should be taken into account self-consistently.

Numerous studies carried out in this direction during the last 10-12 years are concentrated, despite their diversity, around quite narrow range of problems. Without their understanding it is difficult to take one's bearing in the modern stellarator theory. That is why we selected them as a subject for this review.

Numerical methods are playing now a big role in the theory. Though numerical simulations give rather precise information, they entail new problems from interpretation of their results to the questions of their applicability limits and generalization. Analytics, not always pretending on high accuracy, in many cases allows to find simple solutions generalizing and explaining the numerical results, to reveal general dependencies, to restrict and clarify the area of further searches. We, therefore, will lay the main stress on analytics, giving preference to the qualitative description of effects, but also turning, when necessary, to the numerical results.

2. Two-dimensional model of a stellarator

2.1. Two-dimensional equilibrium equation for a stellarator

The basis of the theory of plasma equilibrium in toroidal systems is the equations

$$ \nabla p = [jB], \quad (1a) $$

$$ \text{div} \ B = 0, \quad j = \text{rot} \ B. \quad (1b) $$

Here $p$ is the plasma pressure, $j$ is the current density, $B$ is the magnetic field induction. Detailed
substantiation of applicability of Eq. (1a) for the description of nonideal plasma one can find in [35].

For axisymmetric systems, which are dominated by the tokamaks, Eqs. (1) are reduced to the well-known Grad-Shafranov equation [36-38]

\[
\text{div} \frac{\nabla \phi}{r^2} = -4\pi^2 \rho'(\phi) - \frac{FF'(\phi)}{r^2},
\]

where \( F \) and \( \phi \) are the poloidal current and flux, respectively.

\[
B = \frac{1}{2\pi} \left[ \nabla \phi \nabla \zeta \right] + \frac{1}{2\pi} P \nabla \zeta,
\]

(3)

\( \zeta \) is the usual toroidal angle varying between 0 and 2\( \pi \). Here and in the following \( r, \zeta, z \) are the cylindrical coordinates associated with the main axis of the device, as shown in Fig. 1.

Fig. 1. Coordinates \( r, \zeta, z \).

Toroidal stellarators are essentially three-dimensional systems without symmetry, and no exact equation similar to Eq. (2) can be derived for them from Eqs. (1). Stellarators differ from tokamaks by the geometry of a magnetic field. In addition to the strong toroidal (longitudinal) field \( B_0 \), which is present in tokamaks also, stellarator has a weaker helical field oscillating in \( \zeta \). This is just what makes a difference between stellarators and tokamaks. The ratio \( B/B_0 \) is rather small in conventional stellarators. This allows to use an expansion in this parameter when solving Eqs. (1). Finally, the system of the Eqs. (1) can be simplified yielding the two-dimensional equation

\[
\text{div} \frac{\nabla (\psi - \phi_0)}{r^2} = -4\pi^2 \rho'(\phi) (1 + \Omega^0) - \frac{FF'(\phi)}{r^2},
\]

(4)

which is similar to the analogous Grad-Shafranov equation. The reasons and consequences of distinctions between Eqs. (4) and (2), the meaning of the values in Eq. (4) will be discussed later.

The Grad-Shafranov equation plays a fundamental role in the tokamak theory. It was comprehensively studied and discussed in a lot of works, methods for its solving are well developed [35,39-42]. That is why the similarity of Eqs. (2) and (4) has, besides visible outward elegance which veils the inward harmony in unlike effects, the great practical importance also. Owing to this similarity it is possible to use for solving stellarator problems the whole rich experience of the tokamak theory.

The problem of describing stellarators with the help of two-dimensional equations was discussed in the literature many times. It is possible to indicate, at least, six groups of articles, [43-48], [49,50], [51-54], [55-61], [62,63], [64,65] with derivation of these equations from Eqs. (1) by different manners. All these works are unified, first, by exploiting in diverse extents the so-called stellarator expansion which was originally proposed by Greene and Johnson in 1961 [43,44]. And, which is more important, by the similarity or equivalence of the final results of different approaches. As it was shown in [58], these equations either can be reduced to Eq. (4), or can be derived from (4) at supplementary simplifying assumptions.

Strictly speaking, Eq. (4) is approximate. And though the procedure of deriving Eq. (4) from Eqs. (1), sharpened by the efforts of many theoreticians, guarantees its rather good accuracy, still the final answer on the question of confidence in Eq. (4) belongs to numerical simulations. Comparison of the results of two-dimensional calculations based on Eq. (4) with those of numerical solution of the complete three-dimensional equilibrium problem has shown [66-70] that for stellarators with moderate aspect ratios (\( A \geq 7 \)) Eq. (4) turns out to be quite reliable. For compact stellarators (\( A \approx 4 \)) the accuracy of Eq. (4) can be insufficient, but the results of two-dimensional modeling are qualitative-
The approach based on Greene and Johnson's ideas [43-47], which makes it possible to describe stellarators with the help of two-dimensional equations, proved itself to be efficient and exceptionally fruitful. Actually, the majority of important theoretical results on plasma equilibrium and stability in conventional stellarators was obtained on the basis of simplified two-dimensional models widely used both in analytical and numerical calculations. Transition to the two-dimensional description of stellarators after averaging the initial three-dimensional equations is accompanied by the essential simplification in the formulation of the problems and makes them more simple. Of course, two-dimensional equations themselves, including Eq. (4), give incomplete information about plasma behavior in a stellarator. The price of the deliverance from mathematical difficulties is the loss (in a new description) of the most simple and visual: the true three-dimensional "portrait" of the configuration. Before discussing the main results of the theory let us dwell upon the problem of conformity of the approximate two-dimensional model of a stellarator to its real three-dimensional prototype.

2.2. Comments on stellarator approximation

Magnetic field in conventional stellarators has two components: axisymmetric one, $\vec{B}$, which main part is the toroidal field $B_\phi e\zeta$, and "helical" one, $\vec{B}$, oscillating in $\zeta$:

$$\vec{B} = \vec{B} + \vec{B}_\phi. \quad (5)$$

In a real device the vacuum helical field $\vec{B}_\phi = \nabla \varphi_\phi$ is a set of harmonics:

$$\varphi_\phi = \sum_{m>0,1} \varphi_{m \phi} (\rho \sin (lu - m\zeta + \alpha_{m \phi}). \quad (6)$$

Here $\rho, u, \zeta$ are the quasi-cylindrical coordinates attached to the circular geometrical axis $r = R$ of the device, Fig. 2, $\alpha_{m \phi}$ are the constants characterizing the phase shifts. One harmonic dominates in this set. Its multipolarity $l$ and the number of its periods along the major circumference, $m$, are always included in the main parameters of a stellarator. Usually $l = 2$, sometimes $l = 3$. Correspondingly, the devices are called as $l = 2$ and $l = 3$ stellarators. For a large-aspect-ratio stellarator one can use the cylindrical approximation for the potential $\varphi_\phi$

$$\varphi_\phi = \sum_{m>0,1} B_\phi e_{m \phi} \frac{R}{m} (m \rho / R) \sin (lu - m\zeta + \alpha_{m \phi}). \quad (7)$$

where $B_\phi$ is the strength of the longitudinal magnetic field on the axis, $I_1$ are the modified Bessel functions, $e_{m \phi}$ are the dimensionless amplitude factors.

When a helical magnetic field is imposed on the main toroidal field, magnetic field lines become twisted around the axis, forming the so-called magnetic surfaces. Ascending to the Spitzer's idea [22,4], the problem of creation of nested magnetic surfaces necessary for plasma confinement is the key problem of the stellarator concept. Due to its top priority it has been studied thoroughly and in many details. A lot of serious theoretical works is devoted to this subject, see reviews [9,71], existence of rather good magnetic surfaces at a proper choice of external currents in stellarators is confirmed by the direct experiments [72-80] and numerical calculations [1,2,30,32-34,66-69,81-95]. This justifies the wide use of the model of nested magnetic surfaces in the theory, though sometimes it somewhat idealizes the reality, see Fig. 3. The presence of magnetic islands is the drawback inherent to all stellarators. Fig. 3 shows that it is possible to fight against the islands by rather simple means.

Let us assume that magnetic surfaces in a stellarator are described by the equation $\mathcal{F}(\rho, u, \zeta)$.
and approach based on the expansion in this parameters is called stellarator approximation or stellarator expansion [43-47]. Its detailed analysis, discussion of expediency of one or another constraint on stellarator parameters, several concrete examples of its using in calculations one can find in [61]. Not repeating what is written in [61], let us consider the approximate solution of Eq. (8).

By the analogy to Eq. (5) the function $\Psi$ can be represented as

$$\Psi = \phi (\rho,u) + \bar{\phi} (\rho,u,\zeta),$$

and equation (8) can be decomposed on two parts:

$$\nabla \phi = - \langle \hat{B} \cdot \nabla \bar{\phi} \rangle, \quad \tag{12a}$$

$$\nabla \bar{\phi} = - \hat{B} \cdot \nabla \phi - \hat{B} \cdot \nabla \bar{\phi}. \quad \tag{12b}$$

Here and in the following,

$$\bar{f} = \langle f \rangle = \frac{1}{2\pi} \int_0^{2\pi} f \, d\zeta,$$

$$\bar{f} = f - \langle f \rangle, \quad \bar{f} = \int f \, d\zeta. \quad \tag{13}$$

The smallness of the first two parameters in (9) allows us to disregard the last term in Eq. (12b) and to replace the operator $\hat{B} \cdot \nabla$ in the left-hand side of Eq. (12b) by $B_i \mathbf{e}_z \cdot \nabla$. After that Eq. (12b) is elementary integrated:

$$\bar{\phi} = - \frac{r}{B_i} \hat{B} \cdot \nabla \phi \equiv - \frac{r^2}{RB_o} \nabla \phi, \quad \tag{14}$$

Approximate equality here corresponds to the replacing $B_i$ and $\hat{B}$ by their vacuum values $B_o R / r$ and $B_h = \nabla \phi_h$, respectively, which is quite permissible at $\beta \ll 1$. As a result we get for the function $\Psi$ a simple expression via $\phi$:

$$\Psi (r) = \phi - \frac{r}{B_i} \hat{B} \cdot \nabla \phi = \phi (r - \delta r). \quad \tag{15}$$

Here

$$\delta r = \frac{r}{B_i} \hat{B} \cdot \nabla \phi \approx \frac{r^2}{RB_o} \nabla \phi_h. \quad \tag{16}$$
The subscipt "p" is used to denote the vectors without a toroidal component:

\[ a_p = a - e_z (e_z \cdot a) . \]  

The expression (15) shows how the complete three-dimensional solution \( \mathcal{F} \) can be reconstructed in linear in \( B/B_t \) approximation when function \( \phi (\rho, u) \), characterizing the "averaged" configuration is known. Eq. (15) resolves a principal problem of the matching the results of two-dimensional and three-dimensional calculations. From the practical point of view it is important, for example, to know not only "averaged" image of magnetic surfaces, which can be found after solving Eq. (4), but also their real geometry. It is easy to confirm that coordinates \( r_s \) of the real magnetic surface \( \mathcal{F} = \text{const} \) are related with coordinates \( \bar{r}_s \) of its average image \( \phi = \text{const} \) by the equality

\[ r_s = \bar{r}_s + \delta r. \]  

Indeed, \( r_s \) and \( \bar{r}_s \) must satisfy the equation

\[ \mathcal{F} (r_s) = \phi (\bar{r}_s) = \text{const}. \]  

From here we immediately obtain Eq. (18) if Eq. (15) is taken into account. Let us note that Eq. (18) is also useful as showing the way of finding the geometrical parameters of averaged surfaces \( \phi = \text{const} \), necessary to 2-D calculations, from the results of 3-D calculations.

2.3. Function \( \phi_V \) in the equilibrium equation (4)

The equation (4) differs from the Grad-Shafranov equation in that it has two additional functions, \( \phi_V \) and \( \Omega^\phi \). Let us explain, first, the origin and meaning of \( \phi_V \).

In the preceding section we discussed the solution of Eq. (8), but this treatment was incomplete. One consequence of Eq. (8) was not used. It is Eq. (12a). When \( \tilde{\phi} \) function derived from Eq. (12b) is substituted into this equation, and its right-hand side is calculated (for more details, see [61], p. 313), it finally takes the form

\[ \left\{ \mathbf{B}_p + \frac{1}{2\pi} \left[ \nabla \phi_V \nabla \zeta \right] \right\} \cdot \nabla \phi = 0, \]  

where \( \mathbf{B}_p = \mathbf{B} - B_t e_z \) is the poloidal component of \( \mathbf{B} \). It is just a place where we meet first the function \( \phi_V \), which is the most important "two-dimensional" characteristic of stellarators:

\[ \phi_V = \frac{\pi r^2}{B_t} \left[ \nabla \phi \zeta \right] \cdot \left( \frac{\nabla \phi}{|\nabla \phi|} \right) \zeta = \frac{2\pi r^2}{B_t} (\bar{B}_e, \tilde{B}_e) \zeta. \]  

The vector which is multiplied in Eq. (20) by \( \nabla \phi \) has no toroidal component (no projection over \( \nabla \zeta \)). So, according to Eq. (20), it can be represented as \( C [\nabla \phi \nabla \zeta] \). By definition, it must be divergence-free vector (because \( \text{div} \mathbf{B}_p = 0 \)), thus \( C = C (\phi) \). Up to now \( \phi_V \) was introduced as some arbitrary enough function. Therefore we can put \( C = 1/(2\pi) \). Finally we get as a consequence of Eq. (12a):

\[ \mathbf{B}_p = \frac{1}{2\pi} [\nabla (\phi - \phi_V) \nabla \zeta]. \]  

Now, being related with a magnetic field, the function \( \phi \) entering in Eq. (22) acquires quite definite physical meaning. We shall show that, with accuracy up to insignificant additive constant, it is the external poloidal magnetic flux.

Before this let us note that, despite the fact that Eq. (22) was derived by an expansion method, its form is mathematically correct. Indeed, the axisymmetric poloidal magnetic field \( \mathbf{B}_p \), satisfying the condition \( \text{div} \mathbf{B}_p = 0 \), can be represented as

\[ \mathbf{B}_p = \text{rot} A_T = [\nabla (r A_T) \nabla \zeta]. \]  

where \( A_T = A_T e_z \) is the vector potential. If we put

\[ A_T = \frac{\phi - \phi_V}{2\pi r}, \]  

we get Eq. (22), which becomes an approximation only after the substitution of an approximate value of \( \phi_V \) into it. For example, \( \phi_V \) calculated according to Eq. (21).

Poloidal flux \( \phi_{\text{pol}} \) is defined as the flux embraced by a toroidal magnetic surface as by a ring from the outside, Fig. 4. Two conditions, \( \mathbf{B} \cdot \nabla \mathcal{F} = 0 \) and \( \text{div} \mathbf{B} = 0 \), allow to take for calculating \( \phi_{\text{pol}} \) a "horizontal" partition \( z = \text{const} \), cutting the magnetic surface along the line.
Fig. 4. Magnetic surface and the surface (shaded) the flux through which is called poloidal.

\[ r = \tilde{r}_0 + \delta, \tag{25} \]

where \( \delta = e_r \cdot \delta r, \) see Eq. (16). In this case

\[
\phi_{pol} = \int B_z \, dS = \int_0^{2\pi} \int_{\tilde{r}_0}^{\tilde{r}_0+\delta} B_z \, r \, dr \, d\zeta \\
= 2\pi A_T + \int_0^{2\pi} \int_{\tilde{r}_0}^{\tilde{r}_0+\delta} B_z \, r \, dr \, d\zeta. \tag{26}
\]

Integration of \( B_z \) in the last term of Eq. (26) gives a small correction to \( 2\pi A_T \) (of the order of \( \delta^2 / \zeta / \rho^2 \)). Due to this it is sufficient to retain under the integral only \( B_z \). The difference between the values of \( B_z \) at the contour \( r = \tilde{r}_0 \) and at the distance \( \delta \) from it is not large. So, for the main contribution we get

\[
\int_0^{2\pi} \int_{\tilde{r}_0}^{\tilde{r}_0+\delta} \langle B_z \delta \rangle \zeta = \phi_v. \tag{27}
\]

Thus

\[
\phi_{pol} = 2\pi A_T + \phi_v = \phi. \tag{28}
\]

By definition (23), \( 2\pi A_T \) is the flux of magnetic field through the surface bounded by the circle \( r = \tilde{r}_0 = \text{const} \). In other words, it is the quasitokamak flux of axisymmetric field \( \tilde{B}_z \). The quantity \( \phi_v \) is the poloidal flux of a helical field through the surface bounded by the "wavy" contour lying on the magnetic surface and closing on itself after one transit along it, Fig. 4. Relationship (28) reflects that natural fact that the total poloidal flux \( \phi_{pol} \) in a stellarator is a sum of these two fluxes.

The function \( \phi_v \) describes the averaged magnetic surfaces of a stellarator. In the simplest case, when there is no any control or shaping axisymmetric field, they are described, as it is seen from Eq. (4), by the equation \( \phi_v = \text{const} \). Inserting \( \tilde{B} = B_h = \nabla \phi_h \) with \( \phi_h \) as given by Eq. (6) into the Eq. (21), we obtain an explicit expression for \( \phi_v \):

\[
\phi_v = -\frac{\pi r^2}{B_h} \sum_{\rho = 0, \pm \rho} \frac{d\phi_{im}}{d\rho} \phi_{am} (\rho) \\
\frac{n}{m} \cos \left( (l - n) u + \alpha_{im} - \alpha_{am} \right). \tag{29}
\]

The stellarator helical field should be chosen in such a way that these surfaces in the operating volume should be nested. The presence of large terms proportional to \( \cos (l - n) u \) in Eq. (29) is, generally speaking, undesirable. They are related with the splitting of magnetic surfaces and developing of multi-axis configurations with a complex petal structure [43]. Such terms appear in Eq. (29) only in such a case when there are, at least, two harmonics in Eq. (6) with the same period in \( \zeta \). They are called sometimes as "interacting" or satellite harmonics [96].

The function \( \phi_v \) is defined mainly by the principal harmonic of the helical field: \( l = 2 \) in \( l = 2 \) stellarator and \( l = 3 \) in \( l = 3 \) stellarator. If the amplitudes of "satellites" are not large, then, with accuracy within toroidal corrections, \( \phi_v = \phi_v (\rho) \). Independence of \( \phi_v \) on \( u \) means that average magnetic surfaces are nested toroids with circular cross-sections. It is just a desired geometry at designing stellarators and selecting their parameters. We, therefore, shall restrict ourselves by this practically interesting case when particular tasks will be considered.

Traditionally, stellarators were the large-aspect-ratio systems \( A = R / b \gg 1 \), \( b \) being the minor plasma radius). At a some stretch even the ATF \( A = 7 \) and the CHS with \( A = 5 \), which is the most compact helical system among operating devices, can be attributed to this family. At \( A \gg 1 \) it is possible to replace \( r \) by \( R \) (radius of
geometrical axis) in Eq. (29) and to use the representation (7) for \( \varphi_h \). For a single harmonic of a helical field we shall get then

\[
\varphi_v = \psi_\varphi (\rho) = - \pi R^4 B_\delta \delta_{im} \frac{l^3}{m^3} \frac{dI^2}{d\rho}.
\]  

(30)

If \( \varphi_h \) consists of several noninteracting harmonics, then it is necessary to carry out the summation over \( l \) in Eq. (30).

If \( \varphi_v = \varphi_v (\rho) \), then it is possible to give, in addition to Eq. (30), one another simple expression for this function. In tokamak theory there is one well known formula for the rotational transform \( \mu \):

\[
\frac{1}{\mu} = \oint \frac{B_\delta d\ell}{|\nabla \psi|}.
\]  

(31)

were \( d\ell \) is the length element of the contour \( \phi = \text{const} \) in the cross-section \( \zeta = \text{const} \). It is easy to show that it should be valid for the conventional stellarators also \( \langle \mu = - d\phi / d\Phi, d\Phi \equiv \oint B_\delta dS_\perp \rangle \), which is the toroidal flux between adjacent magnetic surfaces, \( dS_\perp = - d\ell \ d\phi / |\nabla \psi| \). In this case the integral in Eq. (31) should be calculated along the "average" boundary, Fig. 5.

For stellarators with circular vacuum magnetic surfaces it follows from Eq. (31):

\[
\mu_h = - \frac{\varphi_v (\rho)}{2 \pi \rho B_\delta}.
\]  

(32)

Here \( \mu_h \) is the vacuum rotational transform.

Function \( \mu_h \) is one of the main characteristics of stellarators, and it is known for each device. It is not only calculated when they are under elaboration and project \( [1,2,9,30,66-69,75,81,82,85-87,89-94] \), but also is well measured experimentally \( [74,75,77,78] \). At known \( \mu_h \) it is natural to use expression (32) to determine \( \varphi_v \). It, first, liberates one from the calculation of \( \varphi_v \) from the vacuum fields which, in their turn, must be calculated before. Second, the transition from \( \varphi_v \) to \( \mu_h \) makes the formulation of the equilibrium problem more physical and makes easier the interpretation of the results. Finally, it allows to formulate a simple theoretical model of a stellarator because in the majority of cases \( \mu_h \) can be approximated with good accuracy by the dependence

\[
\mu_h = \mu_0 + (\mu_b - \mu_0) \rho^2 / b^2.
\]  

(33)

where \( \rho \), as was defined earlier and indicated in Fig. 2, is the radius counted in the transverse cross-section from the geometrical axis, \( b \) is the minor radius, \( \mu_0 = \mu_h (0), \mu_b = \mu_h (b) \). Sometimes, to guarantee the due regard of peculiar features of a stellarator, it is necessary to add into Eq. (33) the next expansion term proportional to \( \rho^4 \).

By definition, \( \mu_h \) is the integral characteristic of a vacuum magnetic configuration, see Eq. (31), and \( \varphi_v \) can be expressed through \( \mu_h \) only if \( \varphi_v = \varphi_h (\rho) \). This last condition is, to some extent, idealization which, however, is almost always justified. Physically, it means that transverse cross-sections of averaged magnetic surfaces are circular and, besides, that their centers coincide. Real 3-D surfaces must have at that the common axis \( r = R \) and their form at different toroidal angles \( \zeta \) must be the same, Fig. 6. In other words, different transverse cross-sections must differ by orientation only. From the series of configurations which can be produced in one or another stellarator it is always possible to find that one which perfectly satisfies this criterion. Parameters of this configuration should be used as initial ones for determining \( \varphi_v \). The radius \( R \) can differ from the geometrical radius of vacuum chamber, as, for example, in the Uragan-2M \( [97] \). This should be taken into account as an important circumstance at formulation of the problems and analysis of the results.
2.4. Function $\Omega^0$ in the equilibrium equation (4)

Poloidal field $\vec{B}_p$, which can be represented in the form (22), is produced by the toroidal current with the density

$$\vec{j}_K = e_\zeta \cdot \text{rot} \vec{B}_p = -\frac{r}{2\pi} \text{div} \frac{\nabla (\phi - \phi_0)}{r^2}. \quad (34)$$

It is clear that the left-hand side of the two-dimensional equilibrium equation (4) just equals to $2\pi j_K / r$. When $j_K$ is calculated, the second "stellarator" value appears

$$\Omega^0 = \langle \vec{B} \rangle^2 / B_0^2. \quad (35)$$

see [43-47]. In stellarators with "noninteracting" harmonics

$$\Omega^0 = -\frac{1}{2\pi R^2 B_0} \sum_l m_l \left[ 2\phi_l + \rho \frac{d\phi_l}{d\rho} \right], \quad (36)$$

where $\phi_l$ are the functions defined by Eq. (30). In the simplest case, when there is only one harmonic of a helical field, the function $\Omega^0$ is expressed directly through $\mu_h$:

$$\langle \vec{B} \rangle^2 / B_0^2 = m \frac{1}{R^2} \int_0^\rho [\mu_h(x) x^4] \frac{dx}{x^2}. \quad (37)$$

Let us note that $\Omega^0$ is the increasing function of the radius $\rho$. Because of this a straight stellarator has so-called "magnetic hill":

$$\zeta = 0 \quad \zeta = \frac{\Gamma}{4} \quad \zeta = \frac{\Gamma}{2}$$

Fig. 6. Cross-sections of real (a - c) and averaged (d) magnetic surfaces of a configuration with $\phi_v = \phi_v (\rho)$; $\Gamma$ is the helical field period

$\frac{d^2 V_0}{d\phi^2} = \frac{R}{\rho B_0^2} \frac{d\Omega^0}{d\rho} > 0$. \quad (38)

Here $V$ is the volume inside magnetic surface, $\phi$ is the longitudinal (toroidal) magnetic flux. Subscript "0" at $V$ denotes that derivative is taken at $\beta = 0$. The quantity (38) is a part of "magnetic well" (hill) and plays an important role in the stability criteria.

The function $\Omega^0$ enters $j_K$ in combination $\Omega^0 R^2 / r^2$, see the next section. By choosing harmonic contents of the helical field it is possible, in principle, to make that $\Omega^0$ will have a "cosine" component $C_1 (\rho) \cos \mu$ with $C_1 > 0$. Due to this it is possible to reduce the Pfirsch-Schluter current and, as a consequence, to increase $\beta_\text{eq}[96]$.

2.5. Formulation of the equilibrium problems on the basis of two-dimensional equation (4)

Similarity of the Grad-Shafranov equation (2) and two-dimensional equilibrium equation (4) for a plasma in stellarators, which was derived in the frame of stellarator approximation, makes it possible to use for their solving the same methods. In other words, the whole reach experience of the tokamak theory can be utilized in stellarator equilibrium problems. The distinctions between Eq. (2) and Eq. (4) can not put an obstacle for that and can be adequately taken into account in any calculation scheme. The methods for solving the Grad-Shafranov equation are described in details in
a lot of review articles, see [35,39-42]. We, therefore, shall restrict ourselves by considering the specific features of stellarator equilibrium problems only.

To solve equations (2) or (4), it is necessary to specify either \( p(\phi) \) and \( F(\phi) \), entering the right-hand side of equilibrium equations, or some another surface functions which allow to express through them \( p \) and \( F \). For example, to prescribe instead of \( F \) the longitudinal current \( J(\phi) \) or rotational transform \( \mu(\phi) \). Besides, for stellarators must be given \( \psi_V \) and \( \Omega^0 \), both of which can be expressed through vacuum rotational transform \( \mu_{\psi} \) if \( \psi_V = \psi_{\psi}(\phi) \). And, finally, in the simplest case the shape and position of the plasma boundary should be prescribed. At such formulation the problem itself and its solution are called a "fixed boundary equilibrium". If plasma boundary is not known, then we say about free boundary plasma equilibrium. In the latter case it is necessary to worry about matching the internal solution of Eq. (4) with the external magnetic field. Examples of both kinds for stellarators will be discussed below.

Mathematical aspects of different equilibrium problems are discussed in details in [41,42].

Stellarator, in contrast to a tokamak, can operate as a stationary system without longitudinal current. To use explicitly the condition \( J(\phi) = 0 \) in the formulation of the equilibrium problem, one should express \( FF'(\phi) \) in the right-hand side of Eq. (4) through \( p \) and \( J \) with the help of the Kruskal-Kulsrud equation,

\[
p' V' = J' \phi' - F' \phi'.
\]

and integral relationships

\[
J = - \alpha_{23} \phi' + \alpha_{33} \phi',
\]

\[
F = - \alpha_{23} \phi' + \alpha_{33} \phi',
\]

relating the currents and fluxes, which can be easily derived from the equilibrium equations (1), see [61]. Here and in the following, prime means the derivative with respect to arbitrary flux coordinate \( a \), if not shown other argument.

It follows from Eq. (39) that

\[
FF'(\phi) = - \rho'(\phi) F \frac{dV}{d\Phi} - \mu F'(\phi).
\]

where \( \mu \) is the rotational transform:

\[
\mu = - \phi' / \phi'.
\]

For the toroidal flux \( \Phi \) the next equality is valid

\[
\Phi = \frac{1}{2\pi} \int B \cdot V \zeta d\tau.
\]

where integration is performed over the volume inside a magnetic surface. Thus, the value \( V'(\Phi) \) in Eq. (41) can be expressed as

\[
\frac{dV}{d\Phi} = \frac{2\pi}{\langle B \cdot V \zeta \rangle}.
\]

We use here a generally adopted notation of averaging over the layer between nearby magnetic surfaces (magnetic or flux surface averaging):

\[
\langle f \rangle = \frac{dV}{d\Phi} \int f d\tau
\]

It is easy to ascertain that at replacing the variables

\[
r = \bar{r} + \delta r,
\]

which transforms the region of integration in Eqs. (43), (45) into the axially symmetric tube, the volume element \( d\tau \) is related to \( d\bar{\tau} \) in new coordinates by the relationship

\[
d\tau = \left(1 + \text{div} \delta r\right) d\bar{\tau},
\]

thus

\[
\int f(r) d\tau = \int \left[f(\bar{r}) + \text{div} f \delta r\right] d\bar{\tau}.
\]

The latter expressions are valid in a linear approximation in the "distortion" \( \delta r \). That is sufficient for calculating \( \langle B \cdot V \zeta \rangle \) which can be represented as

\[
\langle B \cdot V \zeta \rangle = \frac{F}{2\pi R} \left[1 + \frac{R^2}{r^2} - 1 + \frac{R}{r} \frac{B_z}{B_0}\right].
\]

It was taken into account here that \( B_z = F / (2\pi R) \).
\[ \equiv B_0 r / r. \] Now with the help of the Eq. (48), where \( \delta r \) should be substituted in the form (16), we get

\[ \frac{1}{F} \frac{d\Phi}{dV} = \frac{1}{4 \pi^2 R^2} \left[ 1 + \left\langle \frac{R^2}{r^2} - 1 - Q_0^o \right\rangle_0 \right], \tag{50} \]

where \( Q_0^o \) is the value (35), the brackets \( \left\langle \right\rangle_0 \) denote a "quasitokamak" averaging over the volume between adjacent axisymmetric surfaces \( \phi (r, z) = \text{const} \).

Finally, the equilibrium equation (4) reduces to the form

\[ \text{div} \left( \frac{\nabla (\phi - \phi_V)}{r^2} \right) = -4 \pi^2 \rho' (\phi) \]

\[ \left[ 1 + Q_0^o - \frac{R^2}{r^2} \left\langle \frac{R^2}{r^2} - Q_0^o \right\rangle_0^{-1} \right] + \frac{\mu E}{r^2} \frac{\partial J}{\partial \phi} \] \[ \tag{52} \]

In stellarators with a large aspect ratio, \( A \), the term in brackets \( \left\langle \right\rangle \) in the right-hand side of Eq. (50) is much smaller than unity. In a linear approximation in \( A^{-1} \)

\[ Q \equiv 1 - \frac{R^2}{r^2} + Q_0^o \cong Q_0^o - 2k \rho \cos \mu. \tag{53} \]

At \( A \gg 1 \) it is possible to disregard toroidal corrections in the left-hand side of Eq. (52) and in the last term of its right-hand side. Then integro-differential equation (52) will be reduced to the form

\[ \Delta (\phi - \phi_V) \]

\[ = -4 \pi^2 R^2 \left[ \rho' (\phi) \left[ Q - \left\langle Q \right\rangle \right] + J' (V) \right], \tag{54} \]

where \( \Delta \) is the Laplace operator. The similar equation (in somewhat another form) was derived first by J. M. Greene and J. L. Johnson [43,44].

Specifying the functional form of the pressure \( \rho \) and current \( J \) is the most adequate for stellarators which are able, contrary to tokamaks, to confine a currentless plasma \( (J = 0) \). In evolution problems [35,39-42,98-100] it is more convenient to prescribe instead of \( J (\phi) \) the rotational transform profile (which must be frozen at rapid plasma heating). Formulation of such problems for tokamaks and stellarators is similar in many respects, it relieve us from discussing them here.

3. Similarity of equilibrium configurations

3.1. Structure of 2-D equilibrium equation

Some properties of stellarator equilibrium configurations are determined by the structure of the equilibrium equation itself and are, thus, universal. We shall discuss them here conformably to the currentless plasma equilibrium problem in the conventional large-aspect-ratio stellarators. In essence, we shall speak about scalings which follow from the invariance of 2-D equilibrium equation under some variations of relevant parameters. It is not a difficult problem to find these scalings. It is enough to make Eq. (54) dimensionless.

In stellarators with circular “averaged” vacuum magnetic surfaces the function \( \phi_V \) is expressed through the vacuum rotational transform \( \mu_b \):

\[ \phi_V = -2 \pi B_0 \int \rho \mu_b (\rho) \, d\rho. \tag{55} \]

With account of this relationship we get for the quantity \( \Delta \phi_V \) in Eq. (54)

\[ \Delta \phi_V = \frac{2 \phi_0^s}{B^s} \frac{1}{\rho} \frac{\partial}{\partial \rho} \left[ \rho \mu_b \right] \tag{56} \]

Here \( \phi_0^s \) is the constant defined as

\[ \phi_0^s = -\pi B_0 \mu_b b^2. \tag{57} \]

It is equal to \( \phi_V (b) \) if \( \mu_b = \mu_b = \text{const} \) (which corresponds to shearless stellarator).

After Eq. (56) is substituted into the equilibrium equation (54), it takes the form

\[ b^2 \Delta \frac{S}{2} = \frac{d}{dS} \left[ \beta S \left[ \left\langle R_b Q \right\rangle - \frac{R_b Q}{b} \right] \right] \]

\[ + \frac{1}{\xi} \frac{d}{d\xi} \left[ \xi^2 \mu_b \right] \tag{58} \]

where \( S \) and \( \xi \) are the dimensionless poloidal flux and the radius measured in the transverse cross-section, respectively.
$S = \frac{\phi_v}{\phi_v^0}, \quad \xi = \frac{\rho}{b}$.  

(59)

$\beta_{\text{eq}}$ is the quantity which is traditionally assumed as an estimate,

$\beta_{\text{eq}} = \frac{\mu_b}{b} \frac{b}{R}$.  

(60)

$b$ and $R$ are the minor and major radius of a plasma column, $\mu_b$ is the rotational transform at plasma edge, and

$\beta (S) = \frac{2 \rho (S)}{B_0^2}$.  

(61)

The right-hand side of Eq. (58) is determined completely by the next three functions:

$\beta (S), \frac{\mu_b}{b}, \frac{R}{b}$.  

(62)

The first one characterizes the $\beta$ value and plasma pressure profile. The other two characterize the stellarator magnetic configuration: rotational transform profile and vacuum magnetic field inhomogeneity. Equilibrium equations for configurations with the same sets of parameters (62) are identically the same.

We consider here the stellarators with $\phi_v = \phi_v (\rho)$. For such systems the equality (37) is valid, so the last function in (62) can be written as

$$
\frac{R}{b} \Omega = \frac{R <B^2>_x - 2\xi \cos \rho}{b B_0^2} = \omega^0 \frac{1}{L} \int_0^L \left( \frac{\mu_b}{\mu_b} x'^3 \right) \frac{dx'}{x'^2} - 2\xi \cos \rho.
$$

(63)

where

$$
\omega^0 = \frac{\mu_b m_b}{R}
$$

(64)

is the normalization factor which is equal to $R <B^2>_x / b B_0^2$ in $l = 2$ shearless stellarator ($\mu_b = \mu_b = \text{const}$) at a distance $b$ from its geometrical axis. Correspondingly, $R \Omega / b$ in (62) can be replaced by the constant $\omega^0 / L$. Stated above comes to the statement now that stellarators with identical

$$
\frac{\mu_b}{\mu_b}, \quad \frac{\omega^0}{L}
$$

(65)

are indistinguishable from the point of view of equilibrium equation (54) or (58). At the same $\beta (S) / \beta_{\text{eq}}$ and boundary conditions the equilibrium configurations in such stellarators will be physically similar. Let us remind that here $l$ is the multipolarity of a stellarator helical field, $m$ is the number of periods of this field along the major circumference of the device.

3.2. Comparison of stellarators

Fusion installations are technically complicated and costly systems. They are designed and constructed with account of historically established international collaboration and, therefore, do not reproduce each other. Moreover, each installation has its own specifics related with particular goals of investigations which complement, as a rule, the controlled fusion studies carried out in other laboratories. In such situation the comparison of numerical and experimental results related to different installations becomes a difficult problem. It is seen, at least, from the Table where some information about several conventional stellarators is collected. We have not included there such devices as H-1, TJ-II, W VII-AS and W VII-X because they are the stellarators of somewhat another type. They differ from the conventional ones by, at least, two characteristics: they have not planar, but spatial geometrical axis; their vacuum magnetic surfaces cannot be described by the function $\phi_v$ similar to (55).

Analysis of equilibrium equation (54) given above show that for comparison of stellarators one should use as a basis the parameters (65). It is quite convenient because there are only two quantities. Besides, they are physically distinguished and can fully replace the "natural" parameters $l, m, b, R, B_0, \varepsilon_b = B / B_0$ in the equilibrium problem. If $\mu_b$ is parabolic, as given by Eq. (33), then instead of $\mu_b / \mu_b$ in (65) it is sufficient to specify the ratio $\mu_0 / \mu_b$. Thus, conventional stellarators with large aspect ratio, circular averaged vacuum magnetic surfaces and parabolic $\mu_b$ (it's quite a realistic model) can be represented as two-parametric family: as dots on the plane $\mu_0 / \mu_b, \omega^0 / l$. Stellarators having the same pairs of these values can be called similar or equivalent. The solution $S (r)$ of Eq. (58) for one of them will be valid also for all other stellarators with the same $\mu_0 / \mu_b, \omega^0 / l$.  

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Table Parameters of existing and next-generation stellarators

<table>
<thead>
<tr>
<th>Device</th>
<th>m</th>
<th>b (cm)</th>
<th>R (cm)</th>
<th>μb</th>
<th>ω^0 = μbmb/R</th>
<th>β^0 = μbmb/R</th>
<th>μ0</th>
<th>μb/μ0</th>
</tr>
</thead>
<tbody>
<tr>
<td>Uragan-3</td>
<td>9</td>
<td>13.5</td>
<td>100</td>
<td>0.6</td>
<td>0.729</td>
<td>4.86</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Heliotron E</td>
<td>19</td>
<td>20.2</td>
<td>220</td>
<td>2.5</td>
<td>4.3</td>
<td>56.8</td>
<td>0.5</td>
<td>0.2</td>
</tr>
<tr>
<td>Liven'-2</td>
<td>14</td>
<td>11.5</td>
<td>100</td>
<td>0.78</td>
<td>1.26</td>
<td>7</td>
<td>0.22</td>
<td>0.28</td>
</tr>
<tr>
<td>Liven'-2M</td>
<td>8</td>
<td>19.4</td>
<td>112</td>
<td>0.8</td>
<td>1.11</td>
<td>11.1</td>
<td>0.25</td>
<td>0.31</td>
</tr>
<tr>
<td>ATF</td>
<td>12</td>
<td>30</td>
<td>210</td>
<td>0.95</td>
<td>1.63</td>
<td>12.9</td>
<td>0.35</td>
<td>0.37</td>
</tr>
<tr>
<td>CHS</td>
<td>8</td>
<td>20</td>
<td>100</td>
<td>0.8</td>
<td>1.28</td>
<td>12.8</td>
<td>0.33</td>
<td>0.41</td>
</tr>
<tr>
<td>LHD</td>
<td>10</td>
<td>60</td>
<td>390</td>
<td>1</td>
<td>1.54</td>
<td>15.4</td>
<td>0.5</td>
<td>0.5</td>
</tr>
<tr>
<td>Uragan-2M</td>
<td>4</td>
<td>15</td>
<td>157</td>
<td>0.6</td>
<td>0.23</td>
<td>3.4</td>
<td>0.4</td>
<td>0.67</td>
</tr>
</tbody>
</table>

Equilibrium configurations with the same β (S)/β_eq, similar shape and position of the plasma boundary must be completely equivalent in such stellarators in the terms of quantities (59).

The main characteristic of a stellarator largely than others determining its capabilities and properties of equilibrium configuration is the rotational transform μb. As it is shown above, in equilibrium problem not the absolute value of μb itself is important, but its dependence on the radius ρ, normalized profile μb/μ0. In the terms of μb/μ0 all stellarators with ρb = ρb (from shearless ones (μ0 = μb) to I = 3 (μ0 = 0), are described by the single parameter μ0/μb varying between zero and unity. Here, as earlier, μ0 and μb are the values of μb at the axis and at the distance b from the geometrical axis of a stellarator.

The value ω^0 is considered to be related with the specific limit on plasma pressure at which the equilibrium configuration becomes unstable with respect to the displacements of a plasma column as a whole [45]. It was shown in [55] that this limit lies far beyond the usual equilibrium limit β_eq, but account of \langle B^2 \rangle /B^2 term in the equilibrium equation is, nevertheless, necessary: in stellarators with a shear, β_eq decreases significantly due to this term. This can be expected only if ω^0 are comparable with unity because

$$\frac{1}{2} \int_0^1 \left[ \frac{\mu_b x^2}{\mu_b} \right] dx \leq 1, \quad (66)$$

and just ω^0 determines the order of magnitude of the first term in Eq. (63). At ω^0 ≪ 1 it turns out to be smaller than the second one (−2ξ cosu), its influence on plasma equilibrium becomes insignificant. In such case the equality of values ω^0 is not necessary to consider stellarators as equivalent. It is sufficient to have the same μ0/μb.

A concrete example of such systems is the whole family of shearless stellarators of W VII-A type, see [10]. Indeed, in I = 2 stellarators with one harmonic of a helical field

$$\mu_b = \frac{me^2}{4} \left[ 1 + \frac{1}{2} \left( \frac{mb}{R} \right)^2 \right] \quad (67)$$

It is easy to derive this formula using Eq. (32) and retaining in Eq. (30) two first terms of expansion of the Bessel function I_2. Shear is small if mb/R ≪ 1. Correspondingly, ω^0 ≪ 1 at μ0 ≤ 1. In W VII-A, for example, ω^0 = 0.06. The equilibrium equations for a currentless plasma in shearless stellarators can differ only due to the difference in β (S)/β_eq. The differences in parameters of such installations are not important in the equilibrium problem. Thus it is possible for their analysis to consider the single case with \beta (S)/\beta_eq and plasma shape of interest. In the shearless stellarators μ0/μb = 1. I = 3 stellarators also has one common characteristics: μb/μ0 = 0.72. Though in this case ω^0 can be of the order of unity (in Uragan-3 ω^0 ≡ 0.7), it should be taken into account that in such systems
The first term can give a significant contribution to \( RQ/b \) at the plasma periphery only, besides not always. Thus a lot of \( l = 3 \) stellarators can be almost "similar".

In practice one is enforced to compare the stellarators which are not equivalent in the strict sense of the word, see Table. The priority at such comparison should be given to the comparison of values \( \mu_0/\mu_b \) as the most significant parameters. The proximity of values \( \mu_0/\mu_b \) of different installations means that, independently of relationships between other parameters, equilibrium configurations in these devices should have much in common. At such classification the most "close" of the existing and designed stellarators listed in the Table turn out to be the ATF and CHS, though, at quick glance on the Table, one can get an impression that CHS has much more in common with Liven'-2M. However, an objective assessment reveals that the nearest neighbor for both devices, lying between them, is the USA torsatron ATF. One more result, far from obvious, which yields from our analysis: in the ATF and CHS the absolute values of \( \beta_{eq} \) should be identical. We make this conclusion comparing the pairs of values \( \mu_0/\mu_b \) and \( \beta_{eq}^0 \). Differences in \( \omega_b \) are less important than approximate equality of the values of \( \mu_0/\mu_b \), \( \beta_{eq}^0 \). Thus, the CHS can be regarded (from the point of view of plasma equilibrium) as a smaller physical model of the ATF. Liven'-2 and Liven'-2M with almost the same \( \mu_0/\mu_b \) should be also close to similar, though, according to \([89]\), \( \mu_b \) profile in the Liven'-2M differs slightly from the parabola.

Dimensionless parameters which are discussed here give an adequate basis for the comparison of different installations, for the extension of known results on yet uninvestigated areas, and in some cases for reliable predictions without solving the equilibrium problem. Now the majority of MHD problems is solved numerically and, besides, with fixed values of some parameters of the device. As it is shown above, the results of such calculations can be easily generalized with the help of simple scale transformations.

4. Boundary conditions in equilibrium problems

Equations (2) and (4) are, in fact, the equation

\[
L (rA_t) = - r \text{div} \frac{\nabla (rA_t)}{r^2} = \frac{\partial}{\partial t} \tag{69}
\]

for the toroidal component \( A_t \) of the vector potential of axisymmetric magnetic field (23) with specifically defined right-hand side. Eq. (69) should be solved under conditions

\[
rA_t \to 0 \quad \text{when} \quad r^2 + z^2 \to \infty \quad \text{and when} \quad r \to 0. \tag{70}
\]

For a circular current filament \( j_o = \delta (r - r') \times \delta (z - z') \) of radius \( r' \) with the center at \( z' \)-axis, lying in the plane \( z = z' \), its solution is known (see, for example, \([10]\)):

\[
\frac{\partial}{\partial r} \int_{r}^{r'} \frac{1}{\pi t} \frac{1}{\rho + \rho'} \left[ 1 - \frac{r'^2}{2} \right] K (\rho - E (\rho)) \, dr'. \tag{71}
\]

Here \( K \) and \( E \) are the first and the second kind complete elliptic integrals, respectively.

\[
\rho^2 = \frac{4rr'}{(r + r')^2 + (z - z')^2}. \tag{72}
\]

With the help of \( G \), which is the Green's function of problem (69) with conditions (70), the vector-potential \( A_s = A_se_c \) of the self-field of currents \( j_c \) which intersect region \( S \) in the cross-section \( \zeta = \text{const} \), (the particular solution of Eq. (69)) is represented in the integral form:

\[
rA_s = \int_S G (r, z; r', z') j_c (r', z') \, dr' \, dz'. \tag{73}
\]

Together with the second Green's formula which for \( \zeta \)-independent functions is reduced to the equality

\[
\int_S (\nabla u \cdot \nabla v) \, dr \, dz = \oint_{\partial S} \left[ u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n} \right] \, dl, \tag{74}
\]

the relationship (73) leads to

\[
rA_s = \eta rA_t \tag{75}
\]

\[
= \oint_{r} \left[ A_t (r', z') \frac{\partial G}{\partial n} - \frac{G}{r'} \frac{\partial}{\partial n} (r'A_t) \right] \, dl'.
\]
where $A_T = A_S + A_{ext}$, $A_{ext}$ is that part of $A_T$ which is produced by the toroidal currents external to the contour $\Gamma$ (by all currents flowing outside $S$), \( \partial / \partial n = n \cdot \nabla \), $n$ is the normal to $\Gamma$ in the plane $\zeta = \text{const}$.

$$\eta = \int \frac{1}{r} \frac{\partial G}{\partial n} dl' = \begin{cases} 1, & (r, z) \in S \\ 0.5, & (r, z) \in \Gamma \\ 0, & (r, z) \notin S \cup \Gamma. \end{cases} \tag{76}$$

Substituting $A_T$ in the form (24) into the right-hand side of Eq. (75), we get for the contour $\Gamma_p$ lying on the plasma boundary

$$rA_{pl} - n \cdot r (A_{pl} + A_{ext}) = \int_{\Gamma_p} G \hat{B} \cdot dl - \frac{1}{2\pi} \int_{\Gamma_p} \frac{\phi}{r} \frac{\partial G}{\partial n} dl' - \eta \frac{\phi}{2\pi} \tag{77}$$

Here $A_{pl}$ is that part of $A_T$ which is created by the plasma currents, \( dl = [e_n'] dl' \), $\phi_b = \phi_{ir}$, is the value of $\phi$ at plasma boundary. Making transition from Eq. (75) to Eq. (77) we have taken into account that $\phi_{ir} = \text{const}$.

The formula (77) is equally applicable to tokamaks and stellarators. A stellarator differs from a tokamak in that its averaged magnetic surfaces $rA_T = \text{const}$ do not coincide with “averaged isobars” $\phi = \text{const}$. Because of this, a nontrivial contribution (term with $\phi_{ir}$ in Eq. (77)) appears from the first term in right-hand side of Eq. (75). In a tokamak $\phi_{ir} = 0$. In this case Eq. (77) is the mathematical formulation of the virtual casing principle [102].

The method named as virtual casing principle is based on the simple physical analogy. It is known that the problem of finding the external confining field and self-field of plasma currents can be reduced to the calculation of the field produced by the current

$$i = [Bn]. \tag{78}$$

“flowing” over plasma surface (see, for example, [35,39,71,102]). Here $B$ and $n$ are the true (not averaged) magnetic field at the plasma surface and the normal to it. For a tokamak the calculation of self-field of this current by Eq. (73) immediately leads to Eq. (77). In a stellarator the plasma surface is not symmetrical. Because of this one is forced to calculate the magnetic field of the current (78) with the help of Green’s function of 3-D Laplace equation straightforwardly. Such a method of numerical calculation of the plasma fields in a stellarator was used in [103].

The presence of small parameters permits to solve a lot of stellarator problems by an expansion method. An attempt of the reduction of the virtual casing principle to two-dimensional one was undertaken in [104], where the method of expansion in powers of $\rho$ was used for the plasma description. Later [105] the virtual casing principle was successfully matched with more universal classical stellarator approximation [43-47]. Stated in [105] the general formulation of the reduced two-dimensional virtual casing principle for stellarator is, in essence, the another method of deriving Eq. (77).

The examples of employing Eq. (77) in the equilibrium free-boundary problems for stellarators will be considered below. Applications of Eq. (77) for tokamaks ($\phi_{ir} = 0$) are described in [40-42].

5. Equations for the magnetic surface shift

In the equilibrium theory the simple analytical models are often used, which makes possible to describe a plasma with the help of one or two parameters. The most commonly used is the model of circular shifted surfaces [35,39,61] which incorporates the main effect of finite plasma pressure: the outward shift of magnetic surfaces relative to the curvature center (Shafranov shift).

It is known that noncircular plasma shape in a tokamak is in many respects more preferable than circular one [106,107]. Because of this a lot of present-day tokamaks, and all the more those under project, are designed for operation with a noncircular plasma. It is very likely that this tendency will become dominant in stellarator studies also. The pioneer here is the ATF torsatron [1,80], which flexibility allows to study configurations with noncircular averaged magnetic surfaces.

To be able to analyze such configurations, we must take into account not only the magnetic surface shift, but also, as a minimum, their elongation. By setting, for example, that averaged magnetic surfaces are shifted ellipses:
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\[ r = R - \rho \cos \theta = R + \Delta - a \cos \theta, \]

\[ z = \rho \sin \theta = Ka \sin \theta. \quad (79) \]

Here \( R \) is the major radius of a geometrical axis, \( \Delta \) is the shift of the centers of magnetic surfaces \( a = \text{const} \) relative to this axis, \( K \) is the magnetic surface elongation, see Fig. 7. In the numerical simulations for the explicit parametric specification of the magnetic surface shape it is necessary sometimes to introduce several dozens or even more than hundred harmonics, see [85,86,103,108]. In that sense a limitedness of the model (79) is evident. But for the qualitative description of many effects it proves, however, to be useful.

The functional dependencies \( \Delta (a) \) and \( K (a) \) must be found from the equilibrium equation (4). The equation for \( \Delta \) in a stellarator for the case of circular magnetic surfaces \( (K = 1) \) has been obtained first in [45]. Later it was rederived by other methods, refined and analyzed in [51-56,61,100]. In some cases at that instead of (79) the transformation \( \rho = a + \Delta \cos \theta \) and the expansion in \( \Delta / a \) were used, which is permissible for the description of the outer region of a plasma column.

At the derivation of the equation for \( \Delta \), usually the transition to the curvilinear flux coordinates is performed which entails the necessity to calculate a metric tensor of these variables, see [56]. It is possible, however, to derive this equation without such a transition. It is sufficient to multiply the equation (4) or (52) by \( \cos \theta \) and, after that, to average it over the volume between nearby magnetic surfaces, see Eq. (51). The next easily verified consequences of (79) are needed at that,

\[ \nabla \Delta = \frac{K a}{D} \left[ e_2 \sin \theta - e_1 \cos \theta \right]. \quad (80) \]

\[ \nabla \theta = \frac{1}{D} \left[ e_2 (\cos \theta - \Delta') + e_1 K (1 + 2d) \sin \theta \right]. \quad (81) \]

where \( d = a \Delta' / (2K) \),

\[ D = \left[ (\nabla a \nabla \theta) \cdot e_1 \right]^{-1} \]

\[ = Ka \left[ 1 - \Delta' \cos \theta + 2d \sin^2 \theta \right], \quad (82) \]

and the relationship

\[ \langle f(a, \theta) \rangle_0 V_0 = 2\pi \int_0^{2\pi} f r D \, d\theta. \quad (83) \]

Finally, at \( K = 1 \) one obtains the known Greene-Johnson-Weimer equation which can be written in the form [56],

\[ \left( \mu_0 \Delta \right)' + \frac{1}{\mu_0} \left[ \mu_0^2 a^3 \Delta' \right]' = \frac{2b'(a) a^2 R}{(\mu_0 + \mu_\ell) B_0^2} \left[ 1 + B_0^2 V''(\Phi) \frac{\Delta}{2} \right], \quad (84) \]

where \( V''(\Phi) \) is the quantity (38), \( \mu_\ell = RJ/(a\Phi) \) is the current rotational transform. For a shearless stellarator with \( K = K_0 = \text{const} \) the similar equation at \( J = 0 \) has the form

\[ a \Delta'' + 3 \Delta' = \frac{2b'(a) R}{\mu^2 B_0^2} \frac{4}{3K^2 + 1}. \quad (85) \]

In Eqs. (84) and (85) nonlinear terms are omitted which corresponds to the approximation \( \Delta'^2 \ll 1 \). Both equations are derived under the assumptions \( a / R \ll 1 \), \( |\Delta'| \gg a / R \). It must be noted that in the Eq. (85) \( \mu = \mu_0 = 2K_0 a / (K_0^2 + 1) \), where \( \mu_0 \) is the vacuum rotational transform of a stellarator at \( K = 1 \).

At \( \mu_0 = 0 \), \( V''(\Phi) = 0 \) the equation (84) describes a tokamak. This case was analyzed in details in [35]. As it is seen from Eq. (84), a stellarator has several specific features in compari-
son with a tokamak. First, Eq. (84) has a "good" solution at \( \mu_j = 0 \). It is the consequence of the fact that the vacuum field of a stellarator has a rotational transform, and no longitudinal current is needed for plasma equilibrium. Second, Eq. (84) for a tokamak contains the derivatives of the shift \( \Delta \) only, while for a stellarator with a shear the left-hand side of Eq. (84) contains the shift \( \Delta \) itself. In such stellarators the geometrical axis is a distinct center, and the vacuum magnetic field counteract the plasma to shift relative to this axis. Finally, there is a term with a "magnetic hill" \( V_0'(\phi) \) in the right-hand side of Eq. (84). Usually it is not large. The exception can be only Heliotron E, see Table in the section 3.2. Analysis of the Eq. (84) shows that in the systems with a large magnetic hill the plasma can become unstable at large \( \beta^* \)'s with respect to "rigid-body" displacement [45,48,55,59,109]. There is no common opinion on this subject because the equation (84) is not reliable enough for such predictions. Incorporating of the nonlinear terms in this equation leads to more optimistic conclusions [52,53]. But numerical calculations [110] show that \( V_0'(\phi) \)-driven instability is really possible, but at very large aspect ratios. Speaking about effect of the magnetic hill on equilibrium in existing stellarators, we should mention, as one of its manifestations, the comparatively low value of \( \beta_{st} \) in the Heliotron E at very high \( \beta_{eq} \).

Configurations with elongated cross-sections will be discussed later. But even from Eq. (85) one can see in what respects they can be interesting: with the increase of \( K \) the right-hand side of Eq. (85) decreases, and, correspondingly becomes smaller the shift \( \Delta \) also. In other words, such configurations react on the plasma pressure more weaker than that with \( K = 1 \). And this gives an opportunity to advance to higher \( \beta^* \)'s.

In the \( \ell = 3 \) stellarators \( \mu_b = Ca^2 \). In this case the equation (84) for the currentless plasma \( (\mu_j = 0) \) turns out to be singular at the magnetic axis. The vanishing of \( \mu_b \) at the axis in such systems always displays itself in this manner at excessively simplified description, see [9,96,111-115]. It should be considered, first of all, as an indication on the imperfection of the analytical models used. Instead of this, however, there were attempts to overcome the difficulties of the description of the \( \ell = 3 \) stellarators by another manner. It was stated that in such stellarators the most probable are the pressure distributions of the type \( p_0 \left( 1 - a^a / b^a \right) \) with \( \alpha \geq 3 \) [111-115]. This assumptions "saved" the formula

\[
\dot{j}_c = \frac{2p'(a)}{\mu B_0} \cos \theta, \tag{86}
\]

where \( \mu = \mu_b = Ca^2 \) was substituted. As a matter of fact, to make finite the value \( \dot{j}_c \) and, correspondingly, to avoid vanishing of the denominator in the right-hand side of Eq. (84), such severe restrictions are not necessary. At solving equilibrium problems one should keep in mind that the rotational transform \( \mu \) in stellarator equilibrium configurations can (to say more correct, must) differ from the vacuum one, \( \mu_v \). In those cases when it is really important, one should use instead of Eq. (84) the equation

\[
\left( \frac{a^a \mu (\Delta')^a}{a} \right) + \frac{\Delta}{a^3} \left[ \frac{1}{2} \frac{\mu_b}{B^2} \frac{\mu}{B} V_0''(\phi) - \frac{a^3}{a} (\mu_v - \mu) \right]
\]

\[
= \frac{2p'(a) R}{\mu B^2} \left[ 1 + B^2 V_0''(\phi) \frac{a^3}{a} \right]. \tag{87}
\]

Rotational transform is determined by the geometry of a magnetic field and is very sensitive to its variations. The model (79) just serves for their explicit description with the help of parameters \( \Delta \) and \( K \). Though it is more complicated than the model with \( K = 1 \) widely used in analytics, it allows, nevertheless, to derive a quite simple but illustrative expression for \( \mu \).

6. Changes of the rotational transform profile in stellarators

Rotational transform \( \mu = - \phi' / \Phi' \) can be found from the first formula (40) which is the consequence of the equation \( j = \text{rot} \ B \). The quantities \( \alpha_b \) in Eqs. (40) are expressed in a most simple way through the metric coefficients of the flux coordinates with straight field lines, see [116,61,100,117]. However, the transition to these coordinates itself and intermediate algebra are cumbersome. We, therefore, propose here the other, nontraditional for stellarators, method for calculating \( \mu \) analytically. Its simplicity and clearness do not need any comments.
To calculate $\mu$, it is possible to use the relationship (24) which allows to express $\psi'$ in terms of the longitudinal current $J$ and $\psi_V$:

$$J = \int j \cdot \mathbf{V} \frac{1}{2\pi} \, d\tau = -\int \left( \frac{1}{|V a|} \cdot \frac{\nabla (\psi - \psi_V)}{4\pi^2 r^2} \right) \, dS_n$$

$$= -\frac{V_a}{4\pi^2} \left( \frac{\mathbf{V} \cdot \mathbf{V} (\psi - \psi_V)}{r^2} \right)$$

(88)

Here integration is performed over the axially symmetric "quasi-tokamak" volume, the surface integral is taken over its boundary, the brackets $\langle...\rangle_0$ denote the averaging (51) reducing to the Eq. (83).

Calculations for the model (79) yield

$$\alpha_2 = \frac{V_a}{4\pi^2} \left( \frac{\mathbf{V} \cdot \mathbf{V} \psi_V}{r^2} \right) = \frac{a}{R_c} K^2 + 1 \, \frac{1}{2K} f_{22}$$

(89)

where $R_c = R + \Delta$,

$$f_{22} = \frac{1}{2\pi} \int_0^{\pi} \frac{1 + (K^2 - 1) (K^2 + 1) \cos^2 \theta}{(1 - a/R_c \cos \theta) (1 - \Delta \cos^2 \theta + 2 \sin^2 \theta)} \, d\theta$$

(90)

Let us note that quantity $\alpha_{22}$ does not depend on a helical field and turns out to be the same for stellarators and tokamaks.

If $\psi_V = \psi_V (\rho)$, then

$$V a \cdot \nabla \psi_V = -2\pi B_0 \mu_s \frac{K a}{D} (a - \Delta \cos \theta)$$

(91)

where $D$ is the function (82). Here we have used the fact that

$$\rho \frac{e_\rho}{\mathbf{V}} \cdot \nabla a = \frac{K a}{D} (a - \Delta \cos \theta)$$

(92)

and in accordance with Eq. (32) $\psi_V (\rho)$ has been replaced by $\mu_s$. Now, turning once more to the Eq. (83), we get for the parabolic profile of $\mu_s$:

$$\frac{V_a}{4\pi^2} \left( \frac{\mathbf{V} \cdot \mathbf{V} \psi_V}{r^2} \right)_0 = -2\pi B_0 K a^2 \frac{\mu_0 + \mu_s - \mu_b}{R_c}$$

$$\left[ \frac{a^2 K^2 + 1}{2} + 2 \Delta^2 \right]$$

(93)

Really, the condition $\psi_V = \psi_V (\rho)$ can be valid to within the toroidal corrections. Thus, we disregard them in Eq. (93). Stellarators are the systems with large enough aspect ratios, so no serious mistake arises after that.

The general formula (50) yields in our case (elliptical shifted magnetic surfaces)

$$\frac{1}{F} \frac{d\Phi}{da} = \frac{K a}{R_c} (1 + d)$$

(94)

Here one can assume that $F \approx 2\pi R B_0$.

Inserting the calculated quantities into the Eq. (88), we get finally

$$\mu = \mu_{st} + \mu_b$$

(95)

where

$$\mu_{st} (a) = \frac{2K}{K^2 + 1} \left[ \mu_0 + \frac{\mu_b - \mu_0}{b^2} \right]$$

$$\left[ \frac{a^2 K^2 + 1}{2} + 2 \Delta^2 \right] (1 - \delta)$$

(96)

$$\mu_b (a) = \frac{2K}{K^2 + 1} \frac{R J}{2\pi a^2 B_0} (1 - \delta)$$

(97)

$\delta$ is the function expressed through the derivatives of $\Delta$ and $K$ [117]:

$$\delta = \frac{1}{2} \left[ \Delta^2 + d^2 + \frac{K^2 - 1}{K^2 + 1} \right]$$

$$\left[ \frac{\Delta^2}{2} + d - d^2 \right]$$

(98)

At $\Delta^2 \ll 1$, $|d| \ll 1$ the value $\delta$ is small as compared with unity.

It follows from Eq. (96) that at finite plasma pressure the rotational transform in stellarators with shear increases at the axis owing to its shift. This effect was repeatedly noticed in both numerical and analytical calculations [118,119,52,100]. But only in [120] its potential danger was for the first time clearly pointed out. Oak Ridge theoreticians, developing the project of the ATF torsatron, discovered that at circular "in average" plasma boundary due to $\mu$ increase in the center and accompanying decrease of $\mu$ in the middle part of the plasma column, one of the main advantages of the ATF vacuum magnetic configuration, $\mu$ monotony, can disappear, Fig. 8. Plasma then can become unstable [121]. The authors of [120] not only studied this effect, but, what is more important, proposed a radical means of its suppression: control of the plasma shape. More definitely, elongation of
the plasma cross-section with the help of a quadrupole magnetic field.

The combined effect of finite $\beta$ and "average" elongation of magnetic surfaces $K$ (ratio of semi-axes) on the $\mu$ profile is clearly seen from the formula (96). The term with $\Delta^2$ in Eq. (96) describes the $\mu$ increase at the axis when it is shifted from the initial position. This effect, as we have already mentioned, is essential in stellarators with shear only ($\mu_b \approx \mu_o$). At $\beta$ close to the equilibrium limit, $\beta_{eq}$, the axis shift $\Delta (0)$ can be of the order of $b/2$. In the ATF the value $\mu (0)$ would increase then (for $K (0)$ unchanged) approximately by a factor of 2 in comparison with the initial value $\mu_b$.

To suppress so large $\mu (0)$ increase, it is necessary, as Eq. (96) shows, to elongate significantly the near-axis region of the plasma column. Fortunately, it is not a difficult problem. It turns out that in stellarators with shear large values of $K$ in the center can be achieved at rather moderate elongation of a plasma column $K_b$ [120,122,123]. In a tokamak, on the contrary, plasma center reacts weaker on the boundary elongation. This difference is caused by the fact that in stellarators $\mu$ is minimal at the center, but in tokamaks at the periphery. Let us note that for $K \neq 1$ the factor $2K/(K^2 + 1)$ is always less than unity. Thus, not only elongation ($K > 1$), but also "flattening" (horizontal elongation, $K < 1$) of the magnetic surfaces allow to reduce the value of $\mu (0)$.

We mentioned a shear several times not occasionally. Control of a stellarator configuration with the help of a quadrupole field turns out to be effective just if there is a shear. It is seen from Eq. (96), for example, that plasma elongation can decrease the value of the negative shear developing in a stellarator with $\mu_b = \mu_o$ owing to the Shafranov shift. But to keep then $\mu = \text{const}$ is impossible. As a matter of fact, for such conclusions the knowledge about $K (a)$ behavior is needed. Clear notion about this one can get considering the change of the shape of magnetic surfaces under the action of a quadrupole field.

7. Controlling the stellarator configuration with the help of a quadrupole field

To elongate plasma column in tokamaks and stellarators, one needs a quadrupole field

$$B_\delta = - \frac{B_2}{2b} \left[ \nabla (\rho^2 \cos 2u) e_z \right].$$  \hspace{1cm} (99)

Here $B_2$ is a constant equal to the field strength at the distance $b$ from the axis $\rho = 0$. According to Eq. (28) the function $\phi$ describing the averaged magnetic surfaces is a sum of two magnetic fluxes: averaged flux $\phi_\delta$ of a helical field and poloidal flux of an axially symmetric (nonoscillating in $\zeta$) field. Thus, in stellarators with $\phi_\delta = \phi_\delta (\rho)$ and parabolic profile of $\mu_b$, in the presence of a quadrupole field

$$\phi = - \pi B_0 \rho^2 [\mu_0 + (\mu_b - \mu_0) \frac{\rho^2}{2b^2}$$

$$+ A \frac{B_2}{B_0} \cos 2u].$$  \hspace{1cm} (100)

Here $A = R/b$ is the aspect ratio. Toroidal corrections in Eqs. (99) and (100) are neglected. Expression (100) is valid for a vacuum configuration.

When $B_2 = 0$, the surfaces $\phi = \text{const}$ are circular. Under the action of a quadrupole field
Fig. 9. Cross-sections of averaged magnetic surfaces in \( \ell = 2 \) stellarators with a quadrupole field, \( \mu_0 / \mu = 1/3 \):

- (a) \( B_2 = 0 \)
- (b) \( B_2 = 0.95B_2^c \)
- (c) \( B_2 = 1.35B_2^c \)
- (d) \( B_2 = 2B_2^c \)

Dashed line is the rectangle with a 1:2 ratio of sides.

(99) they become elongated, Fig. 9a, b. In shearless stellarators \((\mu_b = \mu_0)\) at \( B_2 < B_2^c \) where

\[
B_2^c = \frac{\mu_0 B_0}{A}.
\]

(101)

the surfaces \( \phi = \text{const} \) are nested ellipses with the constant ratio of semi-axes

\[
K_0 = \left[ \frac{1 + B_2 / B_2^c}{1 - B_2 / B_2^c} \right]^{1/2}.
\]

(102)

In stellarators with shear, as it is seen from Eq. (100), the influence of a quadrupole field turns out to be stronger at the axis and weaker at the periphery, where the role of the second term in the brackets \([\ldots]\) is significant. The quantity \( K_0 \) in this case is the elongation of near-axis surfaces. Let us note that according to Eq. (96), the rotational transform at the axis decreases to zero with the increase of \( B_2 \) to \( B_2 = B_2^c \):

\[
\mu(0) = \frac{2K_0}{K_0^2 + 1} \mu_0 = \sqrt{1 - (B_2 / B_2^c)^2}.
\]

(103)

At \( B_2 < B_2^c \) the surfaces \( \phi = \text{const} \) remain single-axis, up to the \( K_0 \rightarrow \infty \). At \( B_2 > B_2^c \) the axis \( \rho = 0 \) becomes hyperbolic: the lines \( \phi = \text{const} \) in its vicinity are not closed. In shearless stellarators, \( \phi_r \sim \rho^2 \), no nested surfaces at all would then remain. But in stellarators with shear \( \phi_r \) increases faster than \( \rho^2 \), and because of this the lines \( \phi = \text{const} \), which the field (99) tends to take away from the axis \( \rho = 0 \), close up on themselves in the region where growth of \( \phi_r \) becomes stronger than that of the term with \( B_2 \) in Eq. (100). As a result the configuration with a separatrix looking like figure eight is formed, Fig. 9c, d. It is interesting that even when \( K_0 \) becomes infinite, the elongation of the outer surfaces in stellarators with shear can be rather small.

Indeed, at \( B_2 = B_2^c \) the horizontal \((u = 0)\) and vertical \((u = \pi/2)\) semi-axes \( l_x, l_z \) of the contour \( \phi = \text{const} \) are related, evidently, by the relationship

\[
l_x^2 \left[ 2\mu_0 + (\mu_b - \mu_0) \frac{B_2^c}{2b^2} \right] = (\mu_0 - \mu) \frac{B_2^c}{2b^2}.
\]

(104)

From this it follows that in this case the elongation \( K_0 \) of the surface with \( l_x = b \), which we shall call the boundary surface, is equal to the value \( E \):

\[
E = 1 + \frac{4\mu_0}{\mu_0 - \mu}.
\]

(105)

It can be called the critical or limiting elongation: when \( K_0 < E \) the configuration remains single-axis; when \( K_0 > E \) the doublet-like structure appears, with a two-petal rosette in the center encircled by elongated nested magnetic surfaces, Fig. 9c, d.

In \( \ell = 3 \) stellarators \( \mu_0 = 0 \) and, correspondingly, \( B_2^c = 0, E = 1 \). Formally, it follows from this that near-axis magnetic surfaces in such stellarators must be splitted, turning into the "figures-8" under the action of vanishingly small
quadrupole field. It is the evidence of the topological instability of a central area in $\ell = 3$ systems, and of its inability to resist the external perturbations. The other manifestation of this instability is splitting of the axis and formation of the internal separatrix in $\ell = 3$ stellarators owing to their toroidicity [9,124-127].

$\ell = 2$ stellarators are, in this sense, better. Nonzero value of $\mu_o$ makes them to be more topologically stable, which is rather important argument in their favor when they are compared with $\ell = 3$ systems [16,89].

In thermonuclear studies the best progress is achieved on the tokamaks. Thus, comparison with these systems allows to understand better the price of merits and demerits of other systems. For tokamaks the plasma elongations $K_b = 1.5 - 1.7$ are normal, well-studied range. And the ITER project is aimed at operating with a plasma with $K_b = 2$ [128]. Such "moderate" values of $K_b$, or, probably, higher ones, seem, according to estimates [129,130], to be attractive for stellarators also. However, at typical for stellarators ratios of $\mu_o/\mu_b$ the critical ellipticity $E$ is not large, Fig. 10. In the ATF torsatron, for example, $\mu_o/\mu_b = 0.3/1$, $E = 1.28$. Single-axis configurations with a large elongation of cross-section can be obtained only in stellarators with a very small shear: $E \geq 3$ at $\mu_o/\mu_b \leq 1.05$. The smallness of $E$ in stellarators means that, as it was mentioned earlier, for the effective control of the rotational transform profile [120] small deformations of the plasma boundary are needed. It is, undoubtedly, the positive factor, because no additional serious problems appear, related with the manufacturing the vacuum chamber and the design of a magnetic system.

It should be noted that construction of the chamber and magnetic system of the ATF, which is one of the best of modern stellarators, allow to operate with $K_b$ even higher than $E$, see [1,81]. It gives an interesting opportunity of the experimental studies of stellarator configurations with internal separatrix, Fig. 9c. It is difficult now to give an appraisal to their advantages and disadvantages. But this uncertainty, we believe, is temporary.

Dimensions of the "figure-8" appearing in a stellarator with shear at $B_2 > B_{2s}$ increase rather rapidly with increase of $K_b$. At the separatrix $\psi = 0$. With this condition it is easy to obtain from the Eq. (100) for the height $h$ of the separatrix "petal":

$$h = b \sqrt{2 \frac{B_2 - B_{2s}}{\mu_b / \mu_0 - 1}} \frac{B_{2s}}{B_{2s}^2} = b \sqrt{\frac{K_b^2 - E^2}{K_b^2 + 1}}.$$

It is seen from this that in a stellarator with $\mu_b/\mu_0 = 3$ (ATF type) $h = b$ at $B_2 = 2B_{2s}$, which can constitute only about 10% from $B_b$. Thus, owing to the strong dependence of $h$ on $B_2$ (at $B_2$ close to $B_{2s}$) in stellarators with shear it is possible to elongate "figure-eight" up to large dimensions, Fig. 9d. As a matter of fact, it is possible, at quite realistic values of $B_2$ to produce configurations similar to those studied for many years on tokamaks of the Doublet series [131,132], and quite comparable with them by the main parameters [133,134]. Let us note that previously to produce the doublet-like configurations, the systems based on stellarators with $\ell = 7 - 9$ ("Doublestar") have been proposed [135]. More late calculations [133,134] have shown that the most promising in this respect are $\ell = 2$ and $\ell = 3$ stellarators.

Fields of higher multipolarity than quadrupole one can affect only the outer part of a stellarator plasma column. But quadrupole field "penetrates" up to the axis affecting the whole plasma column. Just so strong means of the plasma control in toroidal systems is a dipole (vertical) field, $B_1$. It should be noted that construction of the chamber and magnetic system of the ATF, which is one of the best of modern stellarators, allow to operate with $K_b$ even higher than $E$, see [1,81]. It gives an interesting opportunity of the experimental studies of stellarator configurations with internal separatrix, Fig. 9c. It is difficult now to give an appraisal to their advantages and disadvantages. But this uncertainty, we believe, is temporary.
Its role is important at high $\beta$’s. The last notion needs to be specified. Thus, before dwelling on the problem of the effect of $B_z$ on plasma equilibrium let us discuss the limitations on $\beta$ following from the equilibrium conditions.

8. Finite pressure effects in shearless stellarators

From the viewpoint of theoretical description the shearless stellarators are the simplest in the stellarator family. A lot of equilibrium problems for such systems can be solved analytically. Their results should be treated with some care because stellarators with shear have their own specifics. At the same time these problems and solutions can serve as illustrative examples showing application of analytical methods in the equilibrium theory and some characteristic features of stellarators.

So, let us consider a shearless stellarator with planar circular axis, large aspect ratio, and circular concentric vacuum averaged magnetic surfaces. The shift of the surfaces at finite plasma pressures is described by the Eq. (84), where the term with a magnetic “hill” can be omitted. For a currentless plasma ($I = 0$) in a shearless stellarator ($I_h = I_{lb} = \text{const}$) it can be easily solved:

$$\Delta' = \frac{1}{\beta_{eq}} \frac{b}{a} \frac{2}{B_0} \left[ (\beta(a) - \bar{\beta}(a)) \right]$$

$$= \frac{b}{a} \frac{\beta(a) - \bar{\beta}(a)}{\beta_{eq}^0}. \quad (107)$$

where $\beta_{eq}^0 = \mu_b^1 b / R$. $\beta(a)$ is the local value of $\beta$, and $\bar{\beta} (a) = 2p / B_0$ is the averaged $\beta$ value with $\bar{p} (a)$ defined as

$$\bar{p} (a) \equiv \frac{2}{a^2} \int_0^a p (a) \, da. \quad (108)$$

Equation (84) is valid when $|\Delta'| \ll 1$. This condition allows to estimate $\beta_{eq}$ (limiting $\beta$), which cannot be exceeded because of the loss of equilibrium. It follows from Eq. (107) that $\Delta' (b) = -\bar{\beta} (b) / \beta_{eq}$, thus, the limiting value $\beta_{eq}$ estimated in the frame of the considered model can not be higher than $\beta_{eq}^0$ at any pressure profiles with $p (b) = 0$.

It can be shown that $\beta_{eq}$ rather strongly depends on the pressure profile. If

$$p = p_0 \left[ 1 - \frac{a^2}{b^2} \right]. \quad (109)$$

then $\bar{p} = p_0 (1 - 0,5 a^2 / b^2)$,

$$\Delta' = -\frac{\bar{\beta}}{\beta_{eq}^0} \frac{a}{b} \quad (110)$$

where $\beta = \bar{\beta} (b)$ is the volume-averaged value of $\beta$. If the limiting value of $\beta$ is supposed to be such that it corresponds to $|\Delta'|_{\text{max}} = 1$, then from Eq. (110) we obtain $\beta_{eq} = \beta_{eq}^0$. Let us note that though this estimate has been derived, formally, beyond the limits of applicability of the model used, it turns out to be, nevertheless, as it follows from numerical calculations [136], rather correct.

For a plasma with a pressure profile

$$p = p_0 \left[ 1 - \frac{a^2}{b^2} \right] \quad (111)$$

we have $\bar{p} = p_0 (1 - a^2 / b^2 + a^4 / (3b^4))$ and, according to Eq. (107),

$$\Delta' = -3 \frac{\bar{\beta}}{\beta_{eq}^0} \frac{a}{b} \left[ 1 - \frac{2}{3} \frac{a^2}{b^2} \right]. \quad (112)$$

In this case (in the contrary to the preceding one) the maximum of the value $|\Delta'|$ lies not at the plasma edge, but at $a^2 / b^2 = 0.5$. Setting it to be equal to unity, we find:

$$\beta_{eq} = \beta_{eq}^0 / \sqrt{2} = 0.7 \beta_{eq}^0. \quad (113)$$

If, on the other hand, we should demand, as usually, the fulfillment of the condition $\Delta' (0) < b / 2$, then instead of estimate (113) we should obtain $\beta_{eq} = 0.5 \beta_{eq}^0$.

Owing to the relative shift of magnetic surfaces at finite plasma pressure, in stellarators with $\mu_b = \text{const}$, called shearless or shear free, a small shear appears. According to Eqs. (96), (98) in this case ($K = \text{const}$, $d = 0$)

$$\mu = -2K K^2 + 1 \mu_b \left[ 1 - \frac{3K^2 + 1}{2 (K^2 + 1)} \frac{\Delta'^2}{2} \right]. \quad (114)$$

At parabolic pressure profile $\Delta' = -a \bar{\beta} / (b \beta_{eq})$ (at any $K$), thus $\mu$ profile also becomes parabolic at $\beta \neq 0$.  

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This effect is unpleasant by two reasons. First, at \( \mu' < 0 \) the ballooning modes can become dangerous \([59,109,137,138]\). Second, decreasing with the pressure increase, \( \mu (b) \) can reach a resonance value \( m/n \) with low \( m \) and \( n \), which is extremely undesirable. Experiments on the W VII-A stellarator have shown \([10,17,139,140]\) that the best plasma confinement is provided at \( \mu \) close to some resonance value \( m/n \) (1/2, 1/3, 2/3), but not coinciding with it. At \( \mu (a) = m/n \) plasma confinement is drastically deteriorated. It follows from Eq. (115) that it is impossible to avoid it: \( \mu (b) \) with \( \beta \) rise can be changed by a factor of 1.5-2 in comparison with the previously selected vacuum value. The elongation of a plasma does not help then.

As it is seen from Eq. (114), \( \mu \)-profile in shearless stellarators must be distorted at any \( \rho (a) \). Impossibility of maintaining \( \mu = \text{const} \) at finite \( \beta \) in conventional shearless stellarators is a serious drawback of these systems. No simple means are seen to fight against it. Partially the problem is solved by transition to more complicated configurations of the W VII-AS \([2,10]\) or W VII-X \([33,34]\) type.

Radial dependence of \( \mu \) is determined by the quantity \( \Delta' (a) \) depending on the pressure distribution, \( \rho (a) \). When \( \rho (a) \) is peaked, \( \Delta' \) is nonmonotonic, see Eqs. (111) and (112). Correspondingly, \( \mu (a) \) also turns out to be nonmonotonic, Fig. 11. Profiles \( \mu (a) \) with a minimum at \( a/b = 0.7 \), similar to that shown on Fig. 11, very often appear in numerical simulations of the equilibrium \([47,48,50,67,93,120,139]\), including those for stellarators with shear. Simple analytics shows that "valley" of \( \mu \) at \( a/b = 0.7 \) should be attributed to the choice of \( \rho (a) \) in the form (111). Correspondingly, in the problems where the deformation of \( \mu \)-profile is essential, it should be kept in mind that it is determined not only by \( \beta \) value, but also by a pressure profile, \( \rho (a) \).

9. \( \beta_{\text{eq}} \) in stellarators

When systems with a magnetic confinement of a plasma are analyzed, always the question about limiting equilibrium plasma pressure appears. It is known that by careful optimization of a configuration (shear + magnetic well) it is possible to provide MHD plasma stability up to the \( \beta \sim \beta_{\text{eq}} \) \([1,51,59,80,109,110,141,142]\). In this case the possibility to attain high \( \beta \) is related with the increase of \( \beta_{\text{eq}} \).

\( \beta_{\text{eq}} \) can be defined as such a value which cannot be exceeded because of inadmissible deterioration of equilibrium configuration. The main causes of limitation of \( \beta_{\text{eq}} \) are such finite-pressure effects as the magnetic axis shift (\( \Delta_{\text{ax}} \)) relative to the plasma boundary (Shafranov shift): the shift of the plasma column as a whole; appearance of island structures at the plasma periphery and stochastization of magnetic lines leading to the reduction of plasma minor radius. At fixed size and position of the last closed magnetic surface (which can be controlled with the help of external fields) the generally accepted criterion is that \( \beta_{\text{eq}} \) is reached when the shift of the axis becomes as large as half of the minor radius: \( \Delta_{\text{ax}}/b = 0.5 \). From the formal viewpoint, it is somewhat artificial limitation, because solutions of the equilibrium equation with \( \Delta_{\text{ax}}/b > 0.5 \) are also possible. Such a choice of a concrete quantitative measure of the distortion of a configuration at high \( \beta \) is dictated partially by practical considerations. Let us note that when

\[
\mu = \frac{2K}{K^2 + 1} \mu_0 \left[ 1 - \frac{3K^2 + 1}{2(K^2 + 1)} \frac{\beta^2}{\beta_{\text{eq}}^2} \right].
\]
$\Delta' \sim a$, see Eq. (110), the equality $\Delta_{ax}/b = 0.5$ is equivalent to the condition $\max | \Delta' | = 1$, which means that near-boundary magnetic surfaces are stuck together at the outward side, and poloidal field in this point becomes infinite. In this case the traditionally defined $\beta_{eq}$ is really a limiting value. We shall discuss here the limitations on $\beta_{eq}$ related only with the axis shift.

Very often for $\beta_{eq}$ estimate in stellarators the expression (60) is used, which can be written in the form

$$\beta_{eq}^0 = \mu_0^*(b) b/R,$$  \hspace{1cm} (116)

to show explicitly the dependence of $\mu_0$ at the plasma boundary on the minor radius $b$. This estimate does not reflect the fact that at the same $\mu_0(b)$ stellarators can differ significantly by the value of the rotational transform at the axis, $\mu_0$. Besides, it does not incorporate the effect of the plasma column shape on $\beta_{eq}$. This estimate can be considered as justified only for a shearless stellarator with circular in average magnetic surfaces, but even then not always, see Eq. (113). In many other cases it turns out to be far from the accurately calculated value $\beta_{eq}$.

For $\ell = 3$ stellarators ($\mu_0 = 0$) more moderate limitations on $\beta_{eq}$ than (60) are known [9,113,143]:

$$\beta_{eq}(0) = \alpha \beta_{eq}^0.$$  \hspace{1cm} (117)

Here $\beta_{eq}(0) = 2p(0)/B_0^2$, and $\alpha$ is a coefficient which value is given usually with a large uncertainty: $\alpha = 0.1 - 0.5$.

For $\ell = 2$ stellarators with shear, occupying intermediate position between $\ell = 3$ and shearless stellarators, the next inequality has been proposed by I. S. Danilkin [144]:

$$\beta_{eq}(0) < \frac{3}{2} \frac{\mu_0/2 + \mu_0}{2A} \frac{\Delta \mu}{A}. $$ \hspace{1cm} (118)

Here $A$ is the aspect ratio, $A$ is a function depending on the $\mu(a)$ profile. As it was noted by the author of [144], (118) should be considered as an upper estimate which is several times higher than really attainable values of $\beta_{eq}(0)$.

The estimate character of the formulae given above is conditioned by the complexity of equilibrium problems for a stellarator, which can be analytically solved in rare, sometimes model cases. Accurately $\beta_{eq}$ can be determined in numerical MHD calculations only. However, surpassing analytics in accuracy, numerical calculations sometimes disguise some characteristic dependencies, because very often they are performed for concrete devices. Before turning to the discussion of their results, let us note that in a currentless stellarator $\beta_{eq}$ must be always proportional to $\beta_{eq}^0$. Indeed, the only parameter characterizing plasma pressure in the equilibrium equation (58) is $\beta(S)/\beta_{eq}^0$. Equilibrium is impossible when this ratio exceeds some critical value. Correspondingly, $\beta_{eq} \sim \beta_{eq}^0$. This conclusion is obvious and does not need the solution of the equilibrium problem.

Proportionality coefficients can differ in different cases by the order of magnitude. They are determined by the parameters of magnetic configurations, pressure and current profiles, permissible shifts of internal magnetic surfaces relative to the boundary and of a plasma column shift as a whole, and by plasma column shape. Some general dependencies reflecting the effect of this factors on $\beta_{eq}$ can be expressed by simple and illustrative formulae. Let us start from the dependence of $\beta_{eq}$ on the rotational transform profile.

Calculations [136] for stellarators with $\mu_0 = 0.95, R/b = 7$ (parameters of the ATF torsatron) and different $\mu_0$ have shown that in such systems at fixed plasma boundary $\beta_{eq}$ dependence on $\mu_0/\mu_0(b)$ is, approximately, linear. At $p'(\phi) = \text{const}$ and circular “in average” plasma column boundary

$$\beta_{eq} \equiv (0.15 + 0.85 \mu_0/\mu_0(b))\beta_{eq}^0.$$  \hspace{1cm} (119)

In stellarators with the same $\beta_{eq}^0$ the value of $\beta_{eq}$ is maximal in shearless and minimal in $\ell = 3$ stellarators. It is related with the fact that in stellarators with shear the internal magnetic surfaces are more “fragile” than in shearless systems (because of the $\mu_0$ decrease to the center). Invariance of the equilibrium equation with respect to such variations of stellarator parameters which keep fixed the values (62), allows to conclude that formula (119), summarizing the results of solving a particular equilibrium problem, must be valid in general case also, for all stellarators with $\phi_v = \phi_v$.
Estimate (119) is applicable when with \( \beta \)-rise a plasma column remains in the same unchanged position. This condition is natural for stellarators with a small gap between plasma and vacuum chamber. If a plasma column may be displaced as a whole relative to the geometric axis, then \( \beta_{eq} \) can be even higher than that given by Eq. (119), see Fig. 12a extracted from Ref. [110]. This conclusion seems to contradict the result of [145] showing that in the ATF torsatron \( \beta_{eq} \) decreases when plasma column is shifted either to the left or to the right relative to some “optimal” position, Fig. 12b. As a matter of fact, there is no contradiction at all: decrease of \( \beta_{eq} \) in the ATF occurs due to the decrease of the plasma cross-section at its shifting relative to the position \( \Delta_b \approx -6\text{cm} \), which illustrates a strong dependence of \( \beta_{eq} \) on the minor plasma radius \( b \). This dependence, not evident on the first glance, can be perceived in both Eqs. (116) and (119), if fast decrease of \( \mu_b(b) \) with \( b \)-decrease is taken into account. Similar sharp drop of \( \beta_{eq} \) is possible also in the cases when \( \beta \)-rise is accompanied by the destruction of external magnetic surfaces [93,146]. There are several possibilities of the magnetic surface destruction in finite-\( \beta \) plasma.

Approximate formula (119) gives \( \beta_{eq} \) estimate for a plasma with a circular averaged cross-section. It is known that in a tokamak it is possible to enhance significantly \( \beta_{eq} \) by plasma cross-section elongation [106,147]. For a rough estimate of this effect the next formula is used sometimes [106,148]:

\[
\frac{\beta_{eq}(K)}{\beta_{eq}(1)} \approx \frac{3K^2 + 1}{4}
\]  

(120)

Tokamaks with a noncircular plasma have other advantages also, which have made them ultimately an indisputable leader in thermonuclear studies. Plasma elongation gives an essential positive effect in stellarators with shear also. It was convincingly shown in Refs. [1,120], which laid a basis of the ATF concept. Effect of plasma column shape on \( \beta_{eq} \) in stellarators has been studied in [123]. Calculations have shown that in stellarators with shear it is possible to achieve \( \beta_{eq} \) increase by a factor of 2-2.5 at rather small elongation of plasma boundary. We are talking about plasma elongation
"in average", which is produced by a quadrupole field. Large gain in $\beta_{eq}$ at comparatively small $K_b$ is related with a strong elongation of internal magnetic surfaces. What could be expected at high $K_b$, it is yet unknown. At $K_b > E$, where $E$ is the critical ellipticity (105), configuration is doublet-like, Fig. 9. Equilibrium problem then becomes more complicated, traditional methods of the equilibrium theory are not quite adequate to solve it.

If we restrict ourselves by single-axis configurations only, then the maximal gain in $\beta_{eq}$ due to elongation can be achieved at $K_b = E$ [136]:

$$\beta_{eq} \approx (0.15 + 2.15 \frac{\mu_{0}}{\mu_{b}}(b)) \beta_{eq}^0.$$  \hspace{1cm} (121)

see Fig. 13. Again, as for a "circular" plasma, in shearless stellarators $\beta_{eq}$ turns out to be higher than in similar stellarators with shear.

The estimate of the gain in $\beta_{eq}$ due to elongation in shearless stellarators can be found analytically. In accordance with the said above, this value will be the upper estimate for $\beta_{eq}$ for elongated configurations in stellarators. Resting on the results of the section 7, it is possible to contend that in shearless stellarators $(\mu = \mu_0)$, at least at low $\beta$, the elongation $K$ of magnetic surfaces should be constant over the radius. It was stated in [130] that in a stellarator with elliptical averaged magnetic surfaces with elongation $K$ constant over the radius, $\beta_{eq}(K)$ is also determined by the approximate equality (120). The evident drawback of the relationship (120) is such that it gives unlimited rise of $\beta_{eq}$ with $K$-rise. Thus, it can be used only for $K$ close to unity.

Effect of plasma ellipticity (elongation) on $\beta_{eq}$ in shearless stellarators can be estimated with the help of the Eq. (85) where the dependence (114) $\mu$ on $K$ must be taken into account. For parabolic plasma pressure (109) in a stellarator with $K = \text{const}$

$$\Delta' = - \frac{\beta_{eq}}{\beta_{eq}(1)} \frac{\mu^2}{\mu^2(K)} \frac{4}{3K^2 + 1}.$$  \hspace{1cm} (122)

Correspondingly,

$$\beta_{eq}(K) = \frac{3K^2 + 1}{4} \left( \frac{2K}{K^2 + 1} \right)^2.$$

and $\beta_{eq}(1) = \beta_{eq}^0 = \mu_0 b / R$.

Dependence (123) is shown in Fig. 14. There for comparison the results of calculations [123] are also shown. Both curves show that $\beta_{eq}(K)$ at first rapidly increases with increasing $K$, then $\beta_{eq}$ growth slows down. It follows from Eq. (123) that the gain in $\beta_{eq}$ greater than by a factor of three due to plasma elongation is impossible. At the same time, one and a half or twofold increase of $\beta_{eq}$ seems to be quite real: it is enough to elongate plasma up to the usual for tokamaks values $K = 1.5 - 2$. In

![Fig. 13. Dependence of $\beta_{eq}$ on $\mu_0 / \mu_b$ in stellarators with circular "in average" (O) and ultimately elongated ($K_b = E$) plasma boundary (●) at $\beta(\phi) = \text{const}$. Straight lines are the dependencies (119) and (121).](image)

![Fig. 14. Dependence of $\beta_{eq}$ on the elongation $K$ of the magnetic surfaces in shearless stellarators: 1 — according to formula (123), 2 — calculated in [123].](image)
stellarators with shear \( \beta_{eq} \) grows with increasing \( K_b \) faster than by formula (123), and the same gain in \( \beta_{eq} \) is achieved at \( K_b \rightarrow \infty \). It should be noted that \( \beta_{eq} \) rather strongly depends on the pressure profile. It can be shown with the aid of Eq. (85) that at more peaked profile, \( p = p_0 (1 - a^2/b^2)^2 \), \( \beta_{eq} \) is approximately 1.4 times lower than that in the case considered.

10. Free boundary plasma equilibrium in stellarators

10.1. Types of problems, their formulations, scheme of solution

Complete equilibrium problem includes, besides solving equations (2) or (4):

a) finding out the external (control) magnetic fields, necessary for the maintaining plasma column in a desired equilibrium position and controlling its shape,

b) prediction of plasma behavior in the external fields (plasma shift, change of its shape with \( \beta \)-rise, etc.).

All this range of problems related with equilibrium of a plasma as a whole is unified in the theory under the topic "free boundary plasma equilibrium". Also is close to them the next problem of magnetic diagnostics:

c) finding out the magnetic fields which are produced by plasma equilibrium currents outside plasma column.

For tokamaks all these problems, appearing naturally owing to the necessity of solving the problem of plasma column equilibrium over major radius (which is impossible in tokamaks without an external vertical field), have been studied long ago and rather thoroughly, see [35]. For stellarators until recently the necessity of their solving was not so obvious. The reason was that in stellarators the problem of plasma confinement is solved in the main by the very method of magnetic configuration production: vacuum magnetic field lines form a family of nested surfaces possessing some "strength margin" against the distortions due to plasma currents. In experiments with low-\( \beta \) plasma this "margin" was sufficient. But whether it will be sufficient for the attainment of those plasma parameters which the stellarator branch of thermonuclear researches is aimed at? This question is important both in principle aspect and in the connection with practical goals fixed for the stellarators ATF, W7-AS and CHS and with the design of the next generation devices. Calculations show that, generally speaking, this margin is insufficient. It is suffice to mention the significant distortion of the rotational transform profile at finite \( \beta \) [120]. Besides, in a stellarator at \( \beta \)-rise the shift of a plasma column as a whole can become intolerably large [149]. To avoid it (and to make then possible the attainment of \( \beta \) close to \( \beta_{eq} \)), it is necessary during the discharge to readjust properly the magnetic field configuration. Thus, at large \( \beta \) the mutually complimentary problems a) and b) becomes crucial for stellarators also.

For their solving it is necessary to turn to the Eq. (77), which can be rewritten in the form

\[
\frac{1}{2\pi} \oint_{r_0} G B_{eq} \cdot \mathbf{d}l - \frac{1}{2\pi} \oint_{r_0} \frac{\Phi_{\infty}}{r} \frac{\partial \Phi}{\partial n} \mathbf{d}l' = \begin{cases} r A_{pl} & \text{in vacuum} \\ -r A_{ext} + \text{const} & \text{in plasma.} \end{cases} \tag{124}
\]

For large-aspect-ratio systems instead of Eq. (71) one can use the next approximate expression for \( G \)

\[
G_1 (r,r') = G_0 - \frac{x + x'}{2R} \left[ G_0 + \frac{R}{2\pi} \right], \tag{125}
\]

which is the expansion of \( G \) in small parameters \( x/R, z/R (t^2 \ll 1) \) with the first two terms left. Here

\[
G_0 = \frac{R}{2\pi} \left[ \ln \frac{8R}{\sqrt{(x-x')^2 + (z-z')^2}} - 2 \right]. \tag{126}
\]

\( x = R - r, R \) is the radius of the circular axis in which vicinity we need to know the behavior of \( G (r,r') \). Let us note that including the first toroidal correction in \( G \) is of principal importance for calculating the integrals in Eq. (124). Simplified "cylindrical" expressions for \( G \), given in well-known articles [43,47], which are equivalent to \( G_0 \), are not adequate for these purposes.

At circular plasma boundary \( r_0 \) the expression (124) can be calculated analytically. The simple substitution

\[
B_{eq} \cdot e_0 |_{r_0} = B_1 + \sum_{n=1}^{\infty} H_n \cos n\alpha,
\]
makes it possible to express the result in the form

\[ f_0 + f_1 \cos \alpha + \sum_{n=2}^{\infty} f_n \cos n\alpha = \left\{ \begin{array}{l} rA_{pl} \quad \text{in vacuum} \\ -rA_{ext} \quad \text{in plasma,} \end{array} \right. \]  

(128)

where \( f_n \) are the functions which are expressed through \( H_n \) and \( \phi_r \):

\[ f_0 = b R B_1 \left[ \ln \frac{\delta R}{R_{max}} - 2 \right] + \frac{\gamma \phi_{r0}}{2\pi} . \]  

(129)

\[ f_1 = b R \xi - \phi_1 \frac{b}{4\pi} \frac{\partial \xi}{\partial b} \]  

- \[ -\frac{b}{2} \left( \ln \frac{\delta R}{R_{max}} - 1 + \frac{\xi}{1/2} \right) . \]  

(130)

\[ f_n = -b R H_n \xi^{n-1} - \phi_n \frac{b}{4\pi n} \frac{\partial \xi^n}{\partial b}, n \geq 2. \]  

(131)

Here \( \xi, \alpha \) are polar coordinates associated with the plasma column center, Fig. 15, \( \xi = \frac{l_{\min}}{l_{\max}}, l_{\min} \) is the minimal value out of \( b \) (minor plasma radius) and \( l \) (current radius), and \( l_{max} \) is the largest one, \( \gamma \) is the value (76).

\[ B_1 = \frac{J}{2\pi b} \]  

(132)

is the field produced by the total longitudinal current \( J \) at the plasma boundary. Details of calculations are given in [149,150]. Let us note that Eq. (128) has been obtained under the next simplifying assumptions: aspect ratio is large, plasma column is “in average” circular. Proceeding further to the analysis of concrete problems, we shall assume, besides this, that \( \phi_r = \phi_r(\rho) \), and \( \mu_b \) is parabolic.

If the external magnetic field is known, then Eq. (124) serves to find the plasma boundary \( \Gamma_p \). Then in addition some property of \( \Gamma_p \) should be given to distinct \( \Gamma_p \) from other magnetic surfaces. For example, the constraint can be imposed that plasma touches a limiter or it is limited by a separatrix. Other restrictions are also possible, see [35,39-42]. It happens often in MHD calculations that, in the contrary, the shape and position of plasma boundary are specified. In this case Eq. (128) allows to find the external field necessary to provide the desired shape of a plasma column and to maintain it in a fixed position. Such problems appear when after studying plasma equilibrium and stability certain optimal configurations are found, and it is necessary to clarify what is really needed to create them.

Analyzing finite-\( \beta \) plasma behavior in external fields, it is natural, first of all, to address the problem of plasma response on the external homogeneous field \( B_1 \), which is the strongest means of equilibrium configuration control.

### 10.2. Effect of vertical field on plasma equilibrium in stellarators

Physical aspects of this problem have been discussed in the article [151], rich in content. There, in particular, the imaginary paradox has been explained: in stellarators vertical field can shift the magnetic surfaces, but no strength is produced by \( B_1 \) on currentless plasma in radial direction. Why then it is possible to stretch or to compress radially the plasma column? The answer of the authors of [151] is the next: when \( B_1 \) is changed, currents are induced in a plasma. Their interaction with the vertical field produces the strength shifting the plasma column. When a new equilibrium state is reached, these currents damp, radial strength vanishes, but the desired result, the shift of the plasma column, is already achieved.

External field is related with the total equilibrium magnetic field \( B \) (more correct, with its axially symmetric component) at the “averaged” boundary \( \Gamma_p \) of the plasma column by the relationship (124), for a “circular” plasma by Eq. (128). In the latter case the main component of the external field turns out to be the homogeneous field

\[ B_1 = B_1^{ext} |_{\rho=0} = \frac{1}{R} \frac{\partial}{\partial r} (rA_{ext}) |_{\rho=0}. \]  

(133)

which is called usually transverse or vertical field. To find the dependence of \( B_1 \) on \( \Delta_b \) and \( \beta \), it is left to express the coefficients of Fourier series (127) in Eqs. (129)-(131) through these values. For \( \phi_r \) at parabolic profile of \( \mu_b \) it can be done rather simply. Suffice it to note that
\[ \rho \mathbf{e}_r = \mathbf{e}_r + \Delta \mathbf{e}_r, \quad (134) \]

see Fig. 15, and \( \phi_V \) is related with \( \mu_b \) by the relationship (55). With the aid of Eqs. (22), (80) and (134) it is easy to confirm that for circular plasma boundary \( (K = 1) \)

\[ \left[ B_r \cdot \mathbf{e}_\alpha \right] |_{r_\alpha} = B_0 \frac{b^2}{r} \left[ \frac{1}{\mu_b} \Delta'(b) \cos \alpha \right. \]

\[ - \left. \mu_b \left( 1 - \frac{\Delta \alpha}{b} \cos \alpha \right) \right|_{r_\alpha}. \quad (135) \]

One can easily find from this the quantities \( H_i \).

Finally, for \( B_\perp \) the next expression is obtained [149]:

\[ B_\perp = -\frac{b}{2R} \left[ B_1 \left( \ln \frac{8R}{b} - \frac{3}{2} \right) \right. \]

\[ - B_0 \left[ \mu \Delta' + \Delta_b \left( \frac{a^2 \mu_c}{b^2} \right) \right] \right|_{r_\alpha}. \quad (136) \]

Here only linear in \( \Delta'(b) \) and \( \Delta_b / b \) terms are retained.

Expression (136) is applicable for both tokamaks and stellarators for "circular" in average plasma column. In tokamaks \( \mu_b = 0 \), \( B_0 \mu_c(b) = B_1 R / b \). In this case Eq. (136) literally coincides with the result of [152], and finally reduces to the expression well-known in the tokamak theory [35,39]

\[ B_\perp = -B_1 \frac{b}{2R} \left( \ln \frac{8R}{b} + \beta_1 + \frac{b}{2} - \frac{3}{2} \right). \quad (137) \]

where \( l_i \) is the specific internal inductance of a plasma column, depending on the longitudinal current distribution, and \( \beta_1 = \beta B_0^2 / B_1^2 \).

We mention here the tokamaks by two reasons. First, in deriving Eq. (136) from Eq. (124) we used models and methods of the tokamak theory, which can serve in many respects as an example for stellarators. It is naturally then to aspire also to the accuracy at least not worse than is needed for tokamaks, and to demand that the result must contain, as a limiting case, the corresponding "tokamak" result. Second, the comparison of stellarators, where rotational transform is created by external fields, with tokamaks, where it is produced due to the longitudinal current flowing through the plasma, allows to understand better the nature of one or another effect in these systems, to see the common features and principal differences.

It is known that in the tokamaks plasma equilibrium is impossible without an external vertical field. It is clearly seen in the Eq. (136) or Eq. (137): \( \ln b > 1.5, \Delta' < 0 \), thus both expressions in the brackets in the right-hand side of Eq. (136) are positive. Moreover, at given \( R, b, \) and current and pressure profiles there is no freedom left in the choice of \( B_\perp \). For stellarators the condition (136) turns out to be less restrictive. Actually, owing to the presence of the additional free parameter, \( \Delta_b \). Also \( B_\perp \) becomes "free". For stellarators one can set simultaneously in Eq. (136) \( B_1 = 0 \) and \( B_\perp = 0 \). This corresponds to current-less plasma equilibrium in a conventional stellarator field. In this case Eq. (136) will give the value of the plasma column shift \( \Delta_b \) due to the finite \( \beta \). Vertical field can be used to suppress it. Let us discuss it in more details.

For the precise determination of \( B_\perp \) from Eq. (136) we need to know \( \Delta' \). To find this quantity, which implicitly depends on pressure and current profiles, it is necessary to solve equilibrium equation (4), which reduces in the simplest case to the equation (87) for the magnetic surface shift. Some important consequences of Eq. (136) are seen, however, even without its solving: at \( \beta \to \beta_{eq} \) and without the vertical field the plasma column must expand outward significantly; it can be maintained in the initial position with the aid of the external vertical field \( |B_\perp| \leq B^* / 2 \). Here
is the “effective poloidal field” at the plasma boundary corresponding to the rotational transform \( \mu_h(b) \). Indeed, at \( \beta \) close to the equilibrium limit, \( \Delta'(b) \to -1 \), then \( \Delta_b/b \leq 0.5 - 0.25 \). If we demand \( \Delta_b = 0 \), then we obtain in this case \( B_\Delta = -B^*/2 \).

The next simple formula for the currentless plasma in a shearless stellarator can illustrate the above statements:

\[
\frac{\Delta_b}{b} = \frac{B_\Delta}{B^*} - \frac{\Delta'}{2} = \frac{B_\Delta}{B^*} \frac{\beta}{2\beta_{eq}'}.
\]  

(139)

It is obtained from Eq. (136) when the expression (107) for \( \Delta' \) is used. Similar relationship for stellarators with shear looks like

\[
\frac{\Delta_b}{b} = \frac{1}{2 - \mu_0/\mu_b} \left( \frac{B_\Delta}{B^*} - \frac{\Delta'}{2} \right).
\]  

(140)

It shows that in stellarators with larger shear the plasma column is shifted outward weaker at \( \beta \to \beta_{eq} \). This difference is related with more stronger dependence of the magnetic axis shift on \( \beta \) at larger shear, which eventually determines the equilibrium limit \( \beta_{eq} \). The results of \([103]\) shown in Fig. 16 confirm these conclusions.

Finally, one can say that at high \( \beta \) the “strength margin” of a vacuum magnetic configuration is insufficient for adequate plasma confinement: with \( \beta \)-rise the plasma column can strongly move outward (of course, it is valid also for a current-carrying plasma). The effect is more pronounced at higher \( \beta \). It is well seen from analytics and is confirmed by the reliable numerical calculations \([86,103]\), see Fig. 16. Thus at high \( \beta \), the necessity emerges to control stellarator configuration by means of external fields which should react on the variations of plasma parameters. In tokamaks it is done with the help of equilibrium control systems with magnetic diagnostics in the feedback chain \([128,153,154]\). Stellarators designed for operation with high-\( \beta \) plasma should also have such a system.

10.3. Magnetic fields of equilibrium currents

As it was already mentioned, one part of the general problem of free-boundary equilibrium is the finding the self-field of the equilibrium configuration. This field is usually calculated as a field of a dipole current \((86)\), and, besides, the difference between coordinates \(a, \theta \) and \( \rho, \phi \) is neglected in the right-hand side of Eq. \((86)\), see \([111,114,155-157]\). Such a model does not incorporate two factors: magnetic axis shift relative to plasma boundary and the whole shift of a plasma column due to finite pressure of a plasma. As a matter of fact, it corresponds to the approximation \( \beta \ll \beta_{eq} \).

Equations (124) or (128) allow to solve the problem of finding the plasma-produced poloidal field, \( B_{ph} \), self-consistently. Information about its behavior, obtained from these equations, is necessary for the assessment of possibilities of magnetic measurements and for their interpretation.

In a general case

\[
B^* = B_0 \mu_b(b) b / R
\]  

(138)
The function $rA_{\text{pl}}$ is related with parameters of equilibrium configuration through the quantity $B_{\text{pl}}$ entering in Eq. (124), which should be found from Eq. (22) after solving the equation (4). The latter demands the knowledge of $p(\phi), J(\phi)$. If in experiments they are known, then Eq. (141) can serve for the check-up of theory predictions. It is, undoubtedly, an important matter, but one would like to have more: from $B_{\text{pl}}$, which can be measured, to learn something about plasma.

Substitution of Eq. (128) in Eq. (141) gives for a "circular" currentless plasma

$$B^0_{\text{pl}} = B_{\text{pl}} e_0 = B_2 \frac{b^2}{\rho^2} \cos \theta.$$  \hspace{1cm} (142)

Here $\rho, \theta$ are the "laboratory" coordinates associated with the geometrical axis of a stellarator, Fig. 15,

$$B_2 = \frac{b}{2R} B_0 \left[ \mu A^* + 2 (\mu_\theta - \mu_\phi) \frac{\Delta_b}{b} \right]$$ \hspace{1cm} (143)

is the vertical field produced by plasma currents. At low $\beta$ the right-hand side of Eq. (143) can be transformed with the help of Eq. (87) to

$$B_2 \approx \int_0^{a_0} \frac{a^2}{\rho^2} \frac{B(a)}{B_0} da.$$ \hspace{1cm} (144)

It is, actually, the simplest expression which can be obtained for $B_2$. Generally speaking, it is not rather accurate. It shows that $B_2 = C \beta$. The proportionality coefficient $C$ depends on the profiles $p(a)$ and $\mu(a)$. It is easy to confirm that

$$\frac{B_2^*}{\beta_0^*} \leq - B_2 \leq \frac{B_2^*}{\beta_0^*},$$ \hspace{1cm} (145)

where $\beta_0$ is the $\beta$ value at the axis of a plasma column, and $\beta$ is the volume-averaged $\beta$. The lower limit in inequality (145) is reached at $\mu = \mu_\theta$, that is in a shearless stellarator; and the upper one in $\ell = 3$ stellarator ($\mu = \mu_\phi a^2/b^2$). However, the most interesting is the intermediate case, $\ell = 2$ stellarators such as Heliotron E, Livent'2, ATF, CHS. For such systems from known $B_2$ one can get only approximate estimate of $\beta$. Or to make qualitative assessment of the dependence $p(a)$, if, besides $B_2$, $\beta$ is also known.

Let us note that, in principle, the value of $B_2$ can achieve the level of one-two percent of $B_0$ or even slightly higher. It is seen from (145). Such a field can be reliably measured.

11. Problems of magnetic diagnostics in stellarators

Magnetic measurements are invariably an important part of plasma diagnostics in tokamaks and stellarators. In stellarators from measured magnetic field the energy content $[80,111,114,115,155-161]$, pressure profile $[115,156,157]$, degree of plasma anisotropy $[155]$, amplitude of Pfirsch-Schluter currents $[111,114,155,157]$, plasma shift $[143]$ are determined or estimated.

In stellarators, possibilities of the magnetic diagnostics based on measuring the poloidal fields, are determined, actually, by the relationship (124), which for a plasma column with circular averaged cross-section takes the form (128). Analytical calculations $[150]$ show and numerical simulations $[103]$ confirm that for circular boundary $\Gamma_b$ it is suffice for the description of $rA_{\text{pl}}$, to keep in the left-hand side of Eq. (128) the first two terms. It is clear, that maximal, what can be then expected at measurements of poloidal fields, is determining (besides, of course, $B_1$) the values $H_1$ and $\phi_1$, see Eq. (127). From these values then $\Delta^*$ at the plasma boundary and $\Delta_0$ could be found. Let us show how it could be done.

11.1. Determination of plasma shift from magnetic measurements

The measurable quantities are the total poloidal magnetic flux $\phi_1$, equal to the sum of the plasma self-field flux and that of external field, and the local values of a total magnetic field. To extract the "plasma" part from the measured signals, it is necessary to know the value of the external field $B_1$. In a general case it is natural to consider $B_1$ as unknown (a noticeable contribution to $B_1$ can be due to image currents also). Thus, there are three unknowns in the problem of magnetic measurements: $H_1, \phi_1$ and $B_1$. Relationship (128) gives all three equations necessary to find them. The first equation is the connection of $B_1$ with $H_1$ and $\phi_1$, Eq. (136). Two other
independent equations can be derived by expressing the measurable magnetic quantities with the aid of Eq. (128).

For the total poloidal flux $\phi_\pi$ of magnetic fields nonoscillating in $\zeta$ and for the azimuthal component of a poloidal field $B_\pi = \vec{B}_\pi \cdot \mathbf{e}_\pi$ it follows from Eq. (128)

$$\phi_\pi = 2\pi r (A_\pi + A_{\text{ext}})$$

$$= \phi_1 - 2\pi R_\rho B_\perp \cos \varphi,$$  \hspace{1cm} (146)

$$B_\perp = B_1 \frac{\rho}{\rho} + B_1 \cos \varphi,$$  \hspace{1cm} (147)

where $\phi_1 = 2\pi f_0(\rho)$, $f_0$ is the function (129),

$$\vec{B}_\perp = B_1 - B_\theta \frac{b^2}{\rho^2} + B_1 \frac{\Delta b}{\rho^2}$$

$$+ B_1 \frac{b}{2R} \left[ \ln \left( \frac{8R}{\rho} - 1 - \frac{1}{2} \frac{b^2}{\rho^2} \right) \right],$$  \hspace{1cm} (148)

$$B_1 = B_1 + B_\theta \frac{b^2}{\rho^2} - B_1 \frac{\Delta b}{\rho^2}$$

$$+ B_1 \frac{b}{2R} \left[ \ln \left( \frac{8R}{\rho} + \frac{1}{2} \frac{b^2}{\rho^2} \right) \right].$$  \hspace{1cm} (149)

$B_\perp$ is the external vertical field, $B_\perp$ is the quantity (143). In this notation the equation (136) takes the form

$$B_1 = B_3 + B_r \frac{\Delta b}{b} - \frac{b}{2R} \frac{3}{2} B_1 \left[ \ln \frac{8R}{b} - \frac{3}{2} \right].$$  \hspace{1cm} (150)

At known plasma toroidal current $J$ (which is measured by the Rogowski coil) and plasma minor radius $b$ there are five unknowns in the latter three equations: $B_1, B_3, B_1, B_3$ and plasma column shift $\Delta b$. It is enough to measure two quantities from this set, then other three can be expressed through them.

Magnetic flux $\phi_\pi$ can be measured (by integration of $d\phi_\pi / dt$) with the help of so-called $\phi$-loops. They are two ring conductors lying in the equatorial plane of the torus or in that parallel to it, one in the inward, and another in the outward side of a plasma column, Fig. 17. In such a way it is possible to determine $B_1(\rho)$, which is the value of the field $B_1$ averaged over the circular ribbon $R - \rho \leq r \leq R + \rho$ in the plane $z = 0$. The value $B_1$ can be measured by local probes. Helical magnetic field does not give any contribution to the poloidal flux measured by the circular $\phi$-loops, but can affect the results of the local measurements. To find $\zeta$-independent components of a magnetic field, it is suffice to add the signals from the similar local probes located over the major circumference at the distance equal to a half-period of a helical field.

If $B_1$ and $B_\perp$ are measured at equal distances $\rho$ from the geometrical axis, then it is possible to find the plasma column shift $\Delta b$ by the formula

$$2 \left( B_r + B_1 \right) \frac{\Delta b}{b} = \vec{B}_\perp + B_1 - \frac{\rho}{b^2} \left( B_1 - B_\perp \right)$$

$$+ B_1 \frac{b}{2R} \left[ \ln \left( \frac{b}{\rho} + \frac{1}{2} \frac{b^2}{\rho^2} \right) \right].$$  \hspace{1cm} (151)

which can be easily derived from Eqs. (148) - (150). A simple expression for $\Delta b$ is obtained also for the case when $B_1$ and $\vec{B}_\perp$ are measured at different distances [150]. The corresponding scheme of positioning the $\phi$-loops and local probes in a stellarator is shown in Fig. 17. Other variants are also possible.

The method discussed is similar to the standard method used in tokamaks [35,39], and, actually, is its generalization. When $B_r = 0$, the relationship (151) exactly coincides with the known result for tokamaks [35,39], which serves as a theoretical basis for the design of the feedback link in the chain of the control system for positioning the plasma column [153,154].

So, without solving the equilibrium problem,
without any information about current and pressure profiles, by means of only simple measurements which became a routine in tokamaks, it is possible to find the plasma column shift in a stellarator. With the help of Eq. (124), generally speaking, it is possible to transfer all the magnetic measurement technique developed for tokamaks [162] on stellarators. At that time the equation (124), being mathematically correct, guarantees good accuracy of the result if \( \phi_v \) and "average" plasma boundary \( \Gamma_p \) are properly determined.

11.2. Theory of diamagnetic measurements in stellarators

Equilibrium currents in a plasma produce not only poloidal, but also self toroidal magnetic field. As a result the magnetic flux \( \Phi \) through the transverse cross-section of a plasma changes on a small value \( \Delta \Phi \). Measurements of \( \Delta \Phi \), called diamagnetic, are an inalienable part of magnetic diagnostics practically in all toroidal installations, see [80,107,111,114,115,155-164].

As a basis for the interpretation of diamagnetic measurements the next formula is usually used

\[
2 \frac{\Delta \Phi}{\Phi_0} = \frac{B_i^2}{B_0^2} - \bar{\beta}.
\]

which is simply derived for a cylindrical plasma column [165] and can be used also for tokamaks with a circular plasma [166,152]. For stellarators it needs to be refined. It became clear after experiments of E. D. Andryukhina and O. I. Fedyanin on the Liven'-2 stellarator [158], when a dependence of \( \Delta \Phi \) on the direction of the longitudinal current (in other words, on the sign of \( B_l \)) was observed.

Calculations show [167,61] that in stellarators with a "circular" plasma

\[
2 \frac{\Delta \Phi}{\Phi_0} = \frac{B_i^2}{B_0^2} - \bar{\beta} + 2 \frac{\Delta \Phi_{at}}{\Phi_0},
\]

where

\[
\Delta \Phi_{at} = -\frac{1}{2RB_0} \int_0^{\rho_c} \frac{d}{d\rho} (\rho^2 \phi_v) \, d\rho
= \frac{\pi}{R} \int_0^\rho \frac{d}{d\rho} \left[ \rho^2 \int_0^\rho \mu_b(x) \, dx \right] \, d\rho.
\]

To make easier the comparison of \( \Delta \Phi_{at}/\Phi_0 \) with other terms in Eq. (153), let us transform the last expression:

\[
\frac{\Delta \Phi_{at}}{\Phi_0} = \frac{B_i}{B_0} \frac{b}{R} \frac{\mu}{\mu_b} \frac{\mu}{\mu_b} \gamma.
\]

Here \( \gamma \) is the function of the order of unity depending only on the profiles of longitudinal current and vacuum rotational transform \( \mu_b \):

\[
\gamma = 2 \int_0^\rho \frac{d}{d\xi} \left[ \xi \int_0^\xi \frac{d}{d\xi} \left( \frac{\xi^2}{\mu_b} \right) \, d\xi \right] \, d\xi.
\]

\( \xi = \rho/b \) is the dimensionless radius in the transverse cross-section, \( j_{st} = J/\pi b^2 \) is the average current density. In some cases \( \gamma \) can be rather easily calculated. For example, for \( j_c = j_0 (1 - \xi^2)^m \)

\[
\gamma = \frac{2}{m + 2} \left[ \frac{\mu}{\mu_b} + \frac{3}{2(m + 3)} \left( \frac{1}{1 - \xi^2} \right) \right].
\]

It is maximal in shearless stellarators and minimal in \( \ell = 3 \) stellarators. More peaked is the current, the smaller is this quantity.

In tokamaks diamagnetic measurements are used to determine \( \bar{\beta} \). In stellarators, judging by Eqs. (153) and (155), at low \( \bar{\beta} \) and rather high longitudinal current (conditions typical for experiments of past years) from measured \( \Delta \Phi \) it is possible to determine the value of \( \gamma \). It could give some idea about current profile, which was mentioned yet in [158]. Of course, feasibility of this depends on whether the contribution of \( \Delta \Phi_{at} \) into the measured value of \( \Delta \Phi \) will be noticeable. It is easy to confirm that

\[
2 \frac{\Delta \Phi_{at}}{\Phi_0} : \bar{\beta} = \frac{\mu_b}{\mu_b} \gamma.
\]

where \( \mu = R B_l/(b B_0) \) is the current rotational transform at plasma boundary. It is seen from this that the last term in Eq. (153) can be considerably larger than the first one. However, even at moderate values of \( \bar{\beta} \) it can be small as compared with the second term because

\[
\bar{\beta} : 2 \frac{\Delta \Phi_{at}}{\Phi_0} = \bar{\beta} : 2 \frac{\mu_b}{\mu_b} \frac{b}{R} \gamma.
\]

Thus, at high \( \bar{\beta} \), when a significant bootstrap current can appear in a plasma, it should be difficult to extract \( \Delta \Phi_{at} \) from \( \Delta \Phi \).

The estimates given above allow to conclude
that contribution of $\Delta \Phi_{st}$ into $\Delta \Phi$ is important only at $\bar{\beta} \ll \bar{\beta}_{st}$. With $\bar{\beta}$-rise it becomes smaller. Then an attractive opportunity of estimating current profile from the measured $\Delta \Phi$ vanishes (or, at least, becomes hardly realizable). But, on the other hand, more reliable becomes the determining of $\bar{\beta}$. Experiments on the Heliotron E on the check-up of accuracy of diamagnetic measurements of $\bar{\beta}$ [159] confirm this statement.

If plasma pressure is anisotropic, then in Eq. (152) $\bar{\beta}$ should be replaced by $\bar{\beta}_{st}$ [35]. To get information about parameters of an anisotropic plasma, magnetic measurements have been used in experiments on the Liven'-2 stellarator [155].

Hot plasma is a perfect conductor, thus at rapid processes magnetic fluxes must be frozen into the plasma. The quantity $\Delta \Phi$ in Eqs. (152) and (153) is the difference $\Phi(b) - \Phi_0(b)$ calculated for a stationary equilibrium state, where $\Phi_0(b) = \int B_0 \, dS_1$ is the magnetic flux of a vacuum toroidal magnetic field through the transverse cross-section of a plasma. The fact that longitudinal flux is frozen into the plasma means that during the transition from one state to another the next value must be conserved:

$$\Phi(b) = \Phi_0(b) + \Delta \Phi$$

$$= \Phi_0(b) \left[ 1 - \frac{\bar{\beta}}{2} + \frac{1}{2} \frac{B_i^2}{B_0} + \frac{\Delta \Phi_{st}}{\Phi_0} \right].$$

In cases when the last two terms here can be neglected, this condition reduces to the equality $b^2 (1 - \bar{\beta}/2) = \text{const}$. From this it follows that invariance of $\Phi$ at rapid change of $\bar{\beta}$ on the value $\delta \bar{\beta}$ is provided due to the change of a plasma minor radius:

$$\frac{\delta b}{b} = \frac{\delta \bar{\beta}}{4}.$$ (161)

With $\bar{\beta}$-rise plasma column expands slightly. The area between the plasma boundary and the diamagnetic loop decreases then on the value $\delta S_1 = -\pi b^2 \delta \bar{\beta}/2$. As a result, the total flux through this loop decreases on $\delta \Phi = -\Phi_0 \delta \bar{\beta}/2$. It is just this value what will be measured. It is clear that in the considered case $\delta \Phi$ is exactly the increment of $\Phi$, that is $\Delta \Phi$. Thus, diamagnetic signal $\Delta \Phi$ does not depend on whether longitudinal flux is frozen into the plasma or not.

But possible freezing of the flux into the external conductors, which displays itself in the small change of a vacuum toroidal field during the discharge, must be taken into account at magnetic measurements without fail [168,169]. Effect of metal surroundings on the measured signal can be, of course, different in different situations. In the Liven'-2, for example, essential was the influence of the vacuum chamber [161]. In general, it should be kept in mind that $\Delta \Phi$ is the "plasma contribution" into the change of the toroidal flux. Besides that the outer sources of a toroidal field can affect the results of measurements.

Let us note that (at least, at small $B_i/B_0$) the formula (153) can be used even for compact stellarators such as CHS, and without any limitations on the plasma shape.

12. MHD plasma stability in stellarators

It is not enough for plasma confinement to fulfill only equilibrium conditions. Equilibrium configuration must be stable. At least, with respect to MHD modes which can destroy it in a very short time.

Ideas, general results of MHD stability theory have entered long ago into the range of customary notions. An excellent textbook for the acquaintance with the subject is the monograph [147]. Detailed analysis of general problems of MHD stability in [147] and in other reviews [170-173,41] allows without dwelling on their discussion to turn our attention directly to their applications for stellarators.

Stability of a plasma with respect to small perturbations of its equilibrium state can be investigated with the help of well-known energy principle [174,41,147,170-173]. According to it, plasma is stable if the potential energy $W$ of the perturbed state is nonnegative. For a plasma with fixed boundary

$$W = \frac{1}{2} \int \left[ \left( Q + \xi \left[ \nabla \times A \right] \right)^2 + \gamma \rho (\nabla \times \xi)^2 - K \xi^2 \right] \, dr,$$ (162)

where $\xi$ is the displacement of a plasma element from the initial equilibrium position, $Q = \text{rot} [\xi \, B]$.
is the perturbation of a magnetic field, \( \xi = \xi \cdot \nabla a, a \) is the arbitrary magnetic surface label, \( \gamma = 5/3 \) is the ratio of specific heats of an ideal gas,

\[
K = 2 \frac{[\mathbf{F} \mathbf{V} a] \cdot (\mathbf{B} \cdot \mathbf{V}) \mathbf{V} a}{|\mathbf{V} a|^2 |\mathbf{V} a|^2}
\]  

(163)

Several other expressions for \( K \) through equilibrium quantities are given in \([61]\).

### 12.1. Sufficient stability criterion

Stability condition \( W \geq 0 \) could be satisfied at arbitrary perturbations \( \xi \) in configurations with \( K < 0 \). This inequality, which has a meaning of a sufficient stability criterion, in tokamaks and conventional stellarators is not satisfied \([175]\). It is easy to confirm \([171,176,177,61]\) that after replacing \( \nabla a / |\nabla a|^2 \) by the vector \( \mathbf{g} \) such as \( \mathbf{g} \cdot \mathbf{V} a = 1 \) in the first term in Eq. (162), the form of Eq. (162) will not be changed, but instead of \( K \) it will contain the value

\[
K_1 = K + X^2 |\nabla a|^2 - \mathbf{B} \cdot \mathbf{V} X,
\]

(164)

where \( X = |\mathbf{g}| \cdot \nabla a / |\nabla a|^2 \). Sufficient stability criterion \( K_1 < 0 \) has been discussed in \([177]\). Its particular case is the well-known Solov’ev criterion \([171,176]\)

\[
\frac{1}{V} [ - P' \Psi'' + J' \Psi'' - F' \Phi'' ] + |\mathbf{g}_a|^2 < 0.
\]

(165)

where \( e_b \) is the first contravariant vector of the Hamada flux coordinates \([178]\) (see also \([61,177]\)), other quantities are the same as in Eqs. (39), (40). For a currentless stellarator the Solov’ev criterion turns out to be nontrivial. As it was shown in \([175]\), in \( \ell = 2 \) stellarator with small shear it can be satisfied at \( \beta \) lower than

\[
\beta_c = \frac{1}{2} \frac{\delta w}{w} \mu^2.
\]

(166)

where \( \delta w / w \) is the relative depth of magnetic well, \( w = 2p + \langle B^2 \rangle \). One cannot obtain large values of \( \beta_c \). But the fact itself that, at least at low \( \beta \), any MHD perturbations can be stable in a stellarator, undoubtedly, deserves mentioning.

### 12.2. Mercier criterion for stellarators

If criterion \( K_1 < 0 \) is not satisfied, it yet does not mean that plasma should be unstable. Because, besides the quantity \(-K\xi^2\) the functional (162) contains also two definitely positive (stabilizing) terms. Their contribution in \( W \), and, finally, the sign of \( W \) essentially depends on the type of a perturbation \( \xi \). The most dangerous are perturbations minimizing the functional \( W \). Finding the minimum of \( W \) is the classical variational problem, which cannot be solved in general due to its 3-D nature. It can be done only for some limited families of plasma perturbations. First of all, for local displacements perturbing a narrow layer of a plasma, with an amplitude almost constant along field lines.

Their stability is determined by the well-known Mercier criterion

\[
\left( \frac{\mu^2}{2} - \langle \gamma s \rangle \frac{V^*}{\phi^2} \right)^2 - \langle \alpha s \rangle \langle K_0 \rangle \frac{V^*}{\phi^2} > 0,
\]

(167)

which derivation with detailed explanations and references of key works can be found in \([172,147,61]\). Here \( \mu \) is the rotational transform, brackets \( \langle \ldots \rangle \) denote the averaging (45),

\[
\gamma s = \frac{\mathbf{j} \cdot \mathbf{B}}{|\mathbf{V} a|^2}, \quad \alpha s = \frac{B^2}{|\mathbf{V} a|^2}.
\]

(168)

In a conventional currentless stellarator with a large aspect ratio and circular shifted magnetic surfaces, see Eq. (79),

\[
\langle \alpha s \rangle \equiv B_0^2, \quad \langle \gamma s \rangle \equiv 2p' \Delta / \mu.
\]

(169)

In this case the Mercier criterion (167) reduces to the inequality \([109,119,138,141]\)

\[
\frac{s^2}{4} + \frac{\mu R}{\mu^2 B_0^2} \left[ \langle V''(\Phi) B_0 a - \langle \mu \phi a^2 \rangle J \rangle \right] > 0.
\]

(170)

where \( s = a \mu / \mu \) is the shear of a magnetic field, \( a \) is the current radius, prime denotes the derivative over \( a \), \( V''(\Phi) \) is the quantity (38). The first term in (170) describes the stabilizing influence of the shear, which role was recognized yet at the dawn of thermonuclear researches \([4,22]\). The second one shows destabilizing influence of the "magnetic hill" which is related with the inhomogeneity of \( B^2 \) due to the
helical field. The third term owes its origin to three factors: toroidicity, shear and magnetic surface shift. In conventional stellarators with shear \( \langle \mu_a a^2 \rangle' > 0 \), hence at natural outward displacements of magnetic surfaces \( (\Delta > 0) \) this term is stabilizing. The shift \( \Delta \) induced by finite pressure of a plasma is more strong with larger \( \beta \). Thus its role in (170) increases with increasing \( \beta \). If then multiplier at \( p' \) in (170) becomes negative, then further rise of \( \beta \) only improves the criterion (170). This effect, which is called plasma self-stabilization [141], in stellarators with shear turns out to be quite strong (in shearless systems it is essentially weaker [179]) and allows to reckon on attainment of high \( \beta \) in these systems [1,66-69]. Theory predictions about plasma self-stabilization in stellarators have been confirmed by experiments on the ATF torsatron [80,180,181].

Plasma column can be shifted relative to the geometrical axis of a stellarator by a vertical field. Numerical results [110,145], in a complete agreement with (170), show the strong influence of such a shift on the plasma stability, see Fig. 12. This effect was observed in experiments on the Heliotron E [182]. Properly shifted magnetic surfaces can be produced in a stellarator with two harmonics of a helical field with the same period in \( \xi \). In this case the essential improvement of the plasma stability can be also achieved [109].

Criterion (170) is good from the viewpoint of its simplicity and clearness, but for the strict analysis of the stability its accuracy is insufficient. In particular, it cannot be applied to the systems with small shear where the self-stabilization of a plasma has been predicted almost 20 years ago [179]. Expressions obtained in [119,141,183], which are similar to (170), but include higher order corrections in \( a/R, \Delta', \beta \), are too complicated to dwell upon them in a brief review.

Mercier criterion (167) contains only quantities characterizing an equilibrium configuration. They can be calculated numerically with a desired accuracy at solving the equilibrium problem. A great number of such calculations has been done in recent years, see [31,68,69,108,184-187]. Let us consider two examples. In [68] Mercier criterion was represented in the form [188]

\[
D_M = D_s + D_w + D_1 + D_s > 0 \quad (171)
\]

and all four terms have been separately calculated and, finally, the value of \( D_M \).

\[
D_s = C_{\mu^2}/4. \quad (172)
\]

\[
D_w = -Cp' \langle \alpha_s \rangle \left[ \frac{V}{\Phi} \right]^2 \left[ \frac{V'}{\Phi} \left( \Phi' \right) \right] + p' \left[ \frac{1}{B^2} \right]. \quad (173)
\]

\[
D_1 = -C_{\mu^2} \left[ \frac{V'}{\Phi^2} \right] \left[ \langle \gamma_s \rangle - \frac{J}{\Phi} \langle \alpha_s \rangle \right]. \quad (174)
\]

\[
D_s = C \left[ \frac{V'}{\Phi^2} \right]^2 \left[ \langle \gamma_s \rangle^2 \right] - \langle \alpha_s \rangle \left[ \frac{(j \cdot B)^2}{B^2 | \nabla a |^2} \right]. \quad (175)
\]

\[
C = C_0 \frac{\Phi(\Phi(b))}{\mu^2 \Phi^2}. \quad (176)
\]

\( C_0 \) is a normalizing factor, \( \Phi \) is the toroidal magnetic flux, \( \Phi(b) \) is the value of \( \Phi \) at the plasma edge. Results of calculations of [68] for currentless plasma in \( \ell = 2 \) compact torsatrons (designated by CTm, \( m \) is the number of helical field periods) with aspect ratio \( A < 5 \) are shown in Fig. 18. Figure 18a shows that the dominant stabilizing contribution into \( D_M \) is associated with a magnetic well, \( D_w \). Effect of the shear, more precisely of the term \( D_s \), is essential only at the plasma periphery. Magnetic well becomes broader with increasing \( \beta \), and also minimum value of \( D_M \) increases. Fig. 18b. Due to this plasma remains stable up to \( \beta = \beta_{eq} \).

Let us note that the term with \( \Delta \) in (170) is (as \( V_\Gamma' (\Phi) \) also) a part of magnetic well. Therefore plasma self-stabilization can be spoken about as about deepening of the magnetic well due to the displacement of magnetic surfaces with \( \beta \)-rise. This effect has been pointed out first in [189]. Such deepening turns out to be strong at a large shear. But in the systems with small shear it is weaker and in some cases it already cannot overcome the destabilizing influence of geodesic curvature of magnetic field lines, which is described by the term \( D_s \) in (171), increasing with \( \beta \)-rise. So
It is known that in tokamaks at high $\beta$ the small-scale perturbations localized on the outer side of the torus where magnetic field line curvature is unfavorable (ballooning modes) can be excited, see [190]. It, naturally, compels researches to clarify how they can show themselves in stellarators. Analytical criterion derived by M. I. Mikhailov [137]

$$\frac{s^2}{2} + \frac{\dot{\rho}_R R}{\mu^2 B_0^2} \left[ V_0'' \left( \Phi \right) B_0^2 a - \frac{\left( \mu a^2 \right)'}{\mu a^2} \Delta \right]$$

$$+ \frac{s}{2} \frac{\left( 2\dot{\rho}_R R \right)^2}{\mu^2 B_0^2} > 0, \quad (177)$$

shows that ballooning modes in conventional stellarators with shear are less dangerous than Mercier modes: at $s > 0$ the last term in (177) is stabilizing. Besides, this criterion, as also (170), explicitly demonstrates the strong self-stabilization of a plasma, or the necessity to have magnetic well in a stellarator. All these important consequences of (177) have been confirmed by the detailed numerical calculations [191-193].

However, even in stellarators with large vacuum shear the sign of $s = \dot{\alpha}/\mu$ can be changed in the central part of a plasma at finite $\beta$, see Fig. 8. Therefore one should not disregard the potential danger of the ballooning instability. This is confirmed also by calculations [94,184] for the torsatrons LHD and ATF. Yet more dangerous the ballooning modes can be in "shearless" systems like Wendelstein and TJ-II. It should be noted also that there are numerical results [214] showing that in some cases ballooning modes can be unstable even at $\mu > 0$. This is in apparent contradiction with criterion (177) which was derived by averaging method in large-aspect-ratio approximation for the case when averaged magnetic surfaces are shifted circles. Because of this limitations it is necessary to check ballooning stability numerically in real devices.

As a basis for the analysis of the ballooning mode stability in stellarators the equations derived in [194] (and later in [195]) are usually used. They are applicable for the toroidal systems of arbitrary geometry and includes finite plasma resistivity. In the simplest case for the ideal plasma they are reduced to the next single equation

$$B \cdot \nabla \left( \frac{k_n^2}{B^2} B \cdot \nabla f \right) + 2 \frac{\left[ B_k \right]}{B^2} k_n p' f = 0, \quad (178)$$

where
\[ k = \left( \frac{B \cdot V}{B} \right) \frac{B}{B} \]  

(179)

is the curvature of magnetic field lines,

\[ k = \frac{[B \cdot V a]}{[V a]^2} + \Lambda V a, \]  

(180)

and \( \Lambda \) is the value defined by the equation

\[ B \cdot V a = - \frac{[B \cdot V a]}{[V a]^2} \cdot \text{rot} \frac{[B \cdot V a]}{[V a]^2}. \]  

(181)

Formulation of the problem of ballooning mode stability in stellarators on the basis of the equation similar to Eq. (178) has been discussed in details in the mentioned above articles [184,191-193]. Criterion of their stability is the absence of the solution of equation (178) localized at a small part of a field line.

When the plasma resistivity is taken into account, some restrictions set by ideal magnetohydrodynamics vanish. Magnetic field in this case is not frozen into the plasma, thus shear stabilizing effect decreases [196]. Calculations [197,198] show that in some stellarators the resistive ballooning modes can be unstable at any \( \beta \), while the ideal modes are excited only at \( \beta \) higher than some critical value. Typical growth rates of the resistive modes are, generally speaking, essentially smaller than those of the ideal modes which can lead to the fast uncontrolled outbursts of particles and energy from the plasma column. Therefore the main attention in the stability analysis is paid usually to the ideal modes. But resistive modes should be reckoned with because they can be one of the reasons of deteriorating their plasma confinement with increasing \( \beta \) [199].

12.4. Stability of nonlocal (low-\( n \)) modes

Up to now we have talked about perturbations with large wave numbers over poloidal and toroidal angles. In the complete formulation the stability problem is much more complicated than at \( n \gg 1 \) (\( n \) is the toroidal number of the mode, \( \xi_{nm} \sim \exp i (M\theta - n\zeta) \)). The presence of small parameters allows to reduce it essentially for the conventional stellarators. For perturbations with a period over \( \zeta \) larger than that of a helical field this simplification is achieved by the toroidal averaging of the integrand in Eq. (162) over the helical field period. Finally, Eq. (162) is reduced to the form [44,121,200]

\[ W = \frac{1}{2} \int [Q^2 - j q \cdot \{Q \cdot \xi \}] \]  

(182)

where

\[ Q = [e \cdot V U], Q_\perp = \text{rot} [\xi \cdot B] \]  

(183)

\( \Omega \) is the quantity (53). It is one of the most important general results of the MHD theory of stellarators, which is widely used in the studies of plasma stability in these systems. As the Greene-Johnson equilibrium equation (54), the functional (182) has been obtained also in the frame of "stellarator approximation" with the expansion in parameters (10).

Besides [44,121,200], the whole series of other articles [49-52,64-66,186,201-206] was devoted to the derivation and analysis of the equations describing low-\( n \) perturbations in stellarators. Despite the difference in various approaches, all they are united by the using of ideology of Greene and Johnson articles [43,44], which laid a basis of the modern stellarator theory.

In a compact form the reduced MHD equations describing the dynamics of large-scale perturbations in stellarators with account of plasma inertia and resistivity can be represented as [49,186,206]

\[ \frac{1}{2\pi} \frac{\partial \phi}{\partial t} = - \eta j \cdot \eta j, \]  

(184)

\[ \frac{\rho_m}{R} \frac{d}{dt} [V \cdot j] = - \eta j \cdot \eta j + [V_\perp \cdot V] \cdot e, \]  

(185)

\[ \frac{d}{dt} \rho = 0. \]  

(186)

Here

\[ \frac{d}{dt} = \frac{\partial}{\partial t} + v_\perp \cdot V, \quad v_\perp = [V_\perp \cdot V]. \]  

(187)

\( \eta \) is the plasma resistivity, \( \rho_m \) is its mass density, \( \nabla_\perp = V - e \cdot (e \cdot V) \). Similar equations have been solved numerically in [50,66-70,121,186,199].
Let us dwell upon some results of these works.

The result of principal significance, in our opinion, is the conclusion of [185], made on the basis of a great number of calculations, that in $\ell = 2$ stellarators in a wide range of parameters the local (Mercier-type) perturbations are more dangerous than low-$n$ internal modes. According to [185], the modes with higher $n$ become unstable at lower $\beta$, and their growth rate increases with increasing $n$ approximately proportional to $\sqrt{n}$. It considerably raises the role of the Mercier criterion. For the cases considered in [185], it turns out, actually, to be the sufficient stability criterion which guaranties the plasma stability with respect to any perturbations (except, maybe, ballooning ones). Of course, it is not a strict statement, but, as it follows from [185], rather fair for a lot of cases of practical interest. The same conclusion follows from the analysis of B. N. Kuvshinov [208], who has shown that equations describing the arbitrary-$n$ perturbations near the resonance magnetic surface in toroidal systems have the same structure, and stability condition reduces in many cases to the Mercier criterion. Proximity of the instability thresholds in $\beta$ of low-$n$ ideal modes and Mercier modes in the ATF torsatron has been mentioned in [68,209] and was confirmed later in [69] for $\ell = 2$ torsatrons with aspect ratios $A \geq 7$ and $10 \leq m \leq 14$ ($m$ is the number of helical field periods). The same has been pointed out also in [210] for the W VII-X stellarator. Results of [69] for the torsatron with $m = 14$, $mac/R = 2.4$, where $ac$ is the minor radius of the helical coils, are shown on the Fig. 20. Except the region where the Mercier criterion is not satisfied and the instability ranges in $\beta$ of the modes with $n = 2$ and $n = 3$, which is shown by thick curves, there are also shown the behavior of $\mu(b)$, $\mu(0)$ and minimal value of $\mu$ as functions of $\beta$. When $\mu(0) > \mu_{\min}$ the $\mu$-profile becomes nonmonotonic, with negative shear in the central part of a plasma column. It negatively affects the plasma stability [69]. As shown in Fig. 20, in this region the $n = 2$ and $n = 3$ modes are unstable at any $\beta$ higher than the threshold value. At the same time the mode with $n/M = 2/2$, which maximum lies in the region where $\mu > 0$, at larger $\beta$ becomes stable again.

It is very important that not only stability boundaries for the modes with different $n$ are close, but also the behavior of these modes is similar, in general. Conclusions about stabilization and self-stabilization of the Mercier modes at outward shifting of the magnetic axis in stellarators with shear are valid as well for low-$n$ modes [66,67,69,182,209,211]. Also for the stability of these modes in conventional stellarators the most favorable is the combination of shear and magnetic well [66,185,200].

From this viewpoint it is interesting to look at the installations which have got only one of these two stabilizing factors. Among them are, for example, the Heliotron E (large shear and magnetic hill) and W VII-X and TJ-II (small shear and magnetic well).

For the Heliotron E the most dangerous is the mode $M/n = 1/1$, which should be unstable at $\beta > \beta_{st} = 1.4\%$ [203]. According to calculations [203], this mode can be partially stabilized and $\beta_{st}$ can be increased by increasing the shear at the $\mu = 1$ magnetic surface with the help of additional toroidal field $\Delta B_0$. The fact that stability of ideal-modes is improved only owing to the shear is confirmed by the stability analysis of the same
(\(M/n = 1/1\)) resistive mode: shear variation practically does not influence this mode in the linear regime. Situation becomes better only for the nonlinearly saturated level of instability. It is interesting to note that plasma self-stabilization in the Heliotron E, discovered in [203], is associated not with magnetic well, but with the shear increase near the \(\mu = 1\) surface, which is caused by the distortion of the \(\mu\)-profile, see section 6, Fig. 8. Let us note that calculations [203] have been performed under assumptions that plasma radius does not change when \(\Delta B_0\) is added to the main toroidal field.

Still the absence of magnetic well is a serious drawback. Calculations [110,207,211] show that creation of magnetic well in the Heliotron E with the help of vertical or quadrupole magnetic fields should essentially enhance its capabilities.

At a certain, carefully optimized choice of parameters the systems with low shear can posses rather good characteristics [31-34,108,210,212]. However, as it is seen from Figs. 18, 19, self-stabilization effect in the TJ-II is much weaker than that in stellarators with shear, and it does not allow to reckon on the advancement to high \(\beta\). In Wendelstein-like stellarators it is necessary to keep strictly the optimal values of \(\mu\), deviation from which is fraught with an essential deterioration of plasma confinement [17,139,140,213]. It is associated, probably, with the fact that at the presence of the shear the unstable large-scale plasma perturbations are localized near the resonance surface, but in shearless systems they seize the wide part of a plasma column [200], which is equivalent to the abrupt loss of stability.

MHD theory up to now does not allow to answer unambiguously the question which stellarator system is better. Rather it shows that their capabilities are still far from being exhausted.

13. Conclusion

For the last 10-12 years a great work has been performed in the stellarator theory resulting in the creation of the effective methods for solving a wide range of problems, essential progress in understanding the main features of a plasma behavior in stellarators, refinement of the "old" theory and development of new ideas. The accuracy and predictive power of the MHD theory of stellarators increased up to the level required by up-to-date experiments and necessary for the reliable physical elaboration of the projects of next generation devices. Qualitatively the changes in the theory can be characterized as a transition to such plasma description when the primary attention is paid to the finite-\(\beta\) effects. Their analysis is the main subject of the present-day studies. That is why they are selected as key issues in the present review.

Stellarators are essentially three-dimensional systems. Any control of stellarator configuration by additional magnetic field such as vertical, quadrupole, hexapole field, or additional helical field changes the last closed magnetic surface. This must be estimated carefully as a prerequisite for subsequent MHD and transport studies. The lack of symmetry eliminates the degeneracy in many problems, makes the stellarator theory more rich and diverse than the tokamak theory, but also, on the other hand, much more complicated. Because of the latter a large part of the stellarator theory is represented by the search for the simplified methods for stellarator description and by the development of numerical codes. Their discussion was not a goal of our review. The reduced two-dimensional equations, on which base the majority of the results discussed here was obtained, are, as accurate numerical simulations show, quite reliable for conventional stellarators. It permitted to omit the substantiation and derivation of many equations (a lot of works are devoted to this) and to concentrate basically on their consequences.

Stellarators are too wide class of systems to give their brief and, at the same time, not very superficial description in the frame of a unified approach. Therefore we restricted ourselves here by considering only conventional stellarators which represent the main branch of stellarator studies. We discussed only fundamental principles of the theory, so some questions were left behind the scope of the review.

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